

# New long time existence results for a class of Boussinesq-type systems

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## Abstract

In this paper we deal with the long time existence for the Cauchy problem associated to some asymptotic models for long wave, small amplitude gravity surface water waves. We generalize some of the results that can be found in the literature devoted to the study of Boussinesq systems by implementing an energy method on spectrally localized equations. In particular, we obtain better results in terms of the regularity level required to solve the initial value problem on large time scales and we do not make use of the positive depth assumption.

**Keywords** Boussinesq systems; Long time existence;

## 1 Introduction

### 1.1 The abcd systems

The following *abcd* Boussinesq systems were introduced in [4] as asymptotic models for studying long wave, small amplitude gravity surface water waves:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + a \varepsilon \operatorname{div} \Delta V + \varepsilon \operatorname{div} (\eta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + c \varepsilon \nabla \Delta \eta + \varepsilon \frac{1}{2} \nabla |V|^2 = 0. \end{cases} \quad (1.1)$$

In system (1.1)  $\varepsilon$  is a small parameter while

$$\begin{cases} \eta = \eta(t, x) \in \mathbb{R}, \\ V = V(t, x) \in \mathbb{R}^n, \end{cases}$$

with  $(t, x) \in [0, \infty) \times \mathbb{R}^n$  are approximations of the free surface of the water and of the fluid's velocity respectively. As it will soon be clearer, we mention that the only values of  $n$  for which (1.1) is physically relevant are  $n = 1, 2$ . The above family of systems is derived from the classical mathematical formulation of the water waves problem: considering a layer of incompressible, irrotational, perfect fluid flowing through a canal with flat bottom represented by the plane:

$$\{(x, y, z) : z = -h\},$$

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where  $h > 0$  and assuming that the free surface resulting from an initial perturbation of the steady state can be described as being the graph of a function  $\eta$  over the flat bottom, the water waves problem is governed by the following system of equations:

$$\left\{ \begin{array}{ll} \Delta\phi + \partial_z^2\phi = 0 & \text{in } -h \leq z \leq \eta(x, y, t), \\ \partial_z\phi = 0 & \text{at } z = -h, \\ \partial_t\eta + \nabla\phi\nabla\eta - \partial_z\phi = 0 & \text{at } z = \eta(x, y, t), \\ \partial_t\phi + \frac{1}{2}(|\nabla\phi|^2 + |\partial_z\phi|^2) + gz = 0 & \text{at } z = \eta(x, y, t) \end{array} \right. \quad (\mathcal{WW})$$

where  $\phi$  stands for the fluid's velocity potential and  $g$  is the acceleration of gravity. The operators  $\Delta$  and  $\nabla$  are taken with respect to  $(x, y)$ . Of course, in many applications the above system of equations raises a significant number of problems both theoretically and numerically. This is the reason why an important number of approximate models have been established, each of them dealing with some particular physical regimes. The *abcd* systems of equations deals with the so called Boussinesq regime which we will explain now. We consider the following quantities:  $A = \max_{x,y,t} |\eta|$  the maximum amplitude encountered in the wave motion,  $l$  the smallest wavelength for which the flow has significant energy and  $c_0 = \sqrt{gh}$  the kinematic wave velocity. The Boussinesq regime is characterized by the parameters:

$$\alpha = \frac{A}{h}, \quad \beta = \left(\frac{h}{l}\right)^2, \quad S = \frac{\alpha}{\beta}, \quad (1.2)$$

which are supposed to obey the following relations:

$$\alpha \ll 1, \quad \beta \ll 1 \text{ and } S \approx 1.$$

Supposing actually that  $S = 1$  and choosing  $\varepsilon = \alpha = \beta$ , the systems (1.1) are derived back in [4]. The unknown functions  $(\eta, V)$  in (1.1) represent the deviation of the free surface from the rest state while  $V$  is an  $O(\varepsilon^2)$  approximation of the velocity  $\nabla\phi$  taken at a certain depth. Actually, the zeros on the right hand side of (1.1) represent the  $O(\varepsilon^2)$  terms neglected in establishing (1.1). The parameters  $a, b, c, d$  are also restricted by:

$$a + b + c + d = \frac{1}{3}. \quad (1.3)$$

Asymptotic models taking into account different topographies of the bottom were also derived, see for instance [9], Section 2, for bottoms given by the surface:

$$\{(x, y, z) : z = -h + \varepsilon S(x, y)\},$$

where  $S$  is smooth enough or [8] for slowly varying bottoms i.e. the function  $S = S(t, x, y)$  depends also on time. A systematic study of asymptotic models for the water waves problem along with their rigorous justification can be found for instance in [11].

The study of the local well-posedness of the *abcd* systems is the subject under investigation in several papers see for instance [4], [5], for space dimension  $n = 1$  or [2], [6] (the BBM-BBM case  $b = d > 0, a = c = 0$ ), [10] (the KdV-KdV case  $b = d = 0, a = c > 0$ ) for dimension  $n = 2$ . In [4] it is shown that the linearized equation near the null solution of (1.1) is well posed in two generic cases, namely:

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0 \quad (1.4)$$

$$\text{or } a = c \geq 0 \text{ and } b \geq 0, \quad d \geq 0. \quad (1.5)$$

It was proved in [7] (see also [11]) that the error estimate between the solution of (1.1) and the water wave system is cumulating in time like  $O(\varepsilon^2 t)$ . Thus, solutions of (1.1) that exist on time intervals of

order  $O\left(\frac{1}{\varepsilon}\right)$  are good approximation for  $(\mathcal{W}\mathcal{W})$  as the error remains of small order i.e.  $O(\varepsilon)$ . Actually, due to the previous mentioned error estimate, on time scales larger then  $O\left(\frac{1}{\varepsilon^2}\right)$  the solution  $(\eta, V)$  stops being relevant as an approximation for the original system. The question of long  $O\left(\frac{1}{\varepsilon}\right)$ -time existence of solutions of (1.1) did not receive a satisfactory answer until in [12] where, the case (1.4) was treated and long time existence for the Cauchy problem was systematically proved, provided that the initial data lies in some Sobolev spaces. The difficulty comes from the lack of symmetry (the  $\varepsilon\eta \operatorname{div} V$  term from the first equation of the  $abcd$ -system) of (1.1). Because of the dispersive operators  $-\varepsilon b \Delta \partial_t + a \operatorname{div} \Delta$ ,  $-\varepsilon d \Delta \partial_t + c \nabla \Delta$ , classical symmetrizing techniques for hyperbolic systems of PDE's fail to be successful.

Global existence is known to hold true for (1.1) in dimension  $n = 1$  for some particular cases: the case

$$a = b = c = 0, \quad d > 0,$$

studied by Amick [1] and Schonbek [13] and in the case

$$b = d > 0, \quad a \leq 0, \quad c < 0,$$

assuming some smallness condition on the initial data, see [5]. In both cases, it is assumed that:

$$\inf_{x \in \mathbb{R}} \{1 + \varepsilon \eta_0(x)\} > 0$$

a condition that makes perfect sense from a physical point of view as  $1 + \varepsilon \eta(t, x)$  represents the total height of the water above the bottom in  $x$  at time  $t$ . The latter case uses the Hamiltonian structure of the system, namely, when  $b = d$  we have:

$$\frac{d}{dt} \left( \int \eta^2 + (1 + \varepsilon \eta) V^2 - \varepsilon c (\partial_x \eta)^2 - \varepsilon a (\partial_x V)^2 \right) dx = 0. \quad (1.6)$$

## 1.2 The main result

The aim of this paper is to generalize most of the results presented in [12], namely, we address the long time existence issue for the general  $abcd$  systems. More precisely we wish to construct solutions for (1.1) for which the time of existence is bounded from below by a  $O\left(\frac{1}{\varepsilon}\right)$ -order quantity. The key ingredient is that we establish energy-type estimations for spectrally localized equations and by doing so we require a lower regularity index than the one used in [12]. Also, we avoid using the non-cavitation condition

$$1 + \varepsilon \eta_0(x) \geq \alpha > 0, \quad (1.7)$$

imposed on the initial datum  $\eta_0$  or the curl free condition on the initial data  $V_0$  used in [6]. As opposed to the method used in [12], ours permits us to treat in a unified manner most of the cases corresponding to the values of the  $a, b, c, d$  parameters. In order to carry on our approach we need some restrictions on the  $a, b, c, d$  parameters. More precisely, we will consider the case (1.4) such that  $b + d > 0$  excluding the following two cases:

$$\begin{cases} a = d = 0, \quad c < 0, \quad b > 0. \\ a = b = 0, \quad c < 0, \quad d > 0. \end{cases} \quad (1.8)$$

In Section 5.1 we put forward the basic ingredients in order to obtain long time existence for the former two cases. In view of (1.3),  $b + d > 0$  is not restrictive as far as  $a, c$  are less or equal to 0.

First, we will slightly change the form of (1.1) noticing that the divergence free part of  $V$  remains constant during time. Indeed, formally, if  $(\eta, \bar{V})$  is a solution of (1.1) with initial data

$$\eta|_{t=0} = \eta_0, \quad \bar{V}|_{t=0} = \bar{V}_0,$$

then

$$\partial_t \bar{V} = -\nabla (I - \varepsilon d \Delta)^{-1} \left( \eta + c \varepsilon \Delta \eta + \varepsilon \frac{1}{2} |\bar{V}|^2 \right), \quad (1.9)$$

and consequently we have that:

$$\partial_t \mathbb{P} \bar{V} = 0,$$

where

$$\mathbb{P} \bar{V} = \mathcal{F}^{-1} \left( \left( I - \sum_{i=1, n} \frac{\xi_i \xi}{|\xi|^2} \right) \mathcal{F}(\bar{V}) \right)$$

is the Leray projector over divergence free vector fields. Thus, we get that:

$$\mathbb{P} \bar{V} = \mathbb{P} \bar{V}_0 \stackrel{not}{=} W.$$

Of course, we have that

$$\operatorname{div} W = 0.$$

By setting

$$\begin{cases} \bar{V}_0 = V_0 + W, \\ \bar{V} = V + W \end{cases}$$

we infer that the system verified by the couple  $(\eta, V)$  is the following:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + a \varepsilon \operatorname{div} \Delta V + \varepsilon W \nabla \eta + \varepsilon \operatorname{div}(\eta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + c \varepsilon \nabla \Delta \eta + \varepsilon \frac{1}{2} \nabla |W|^2 + \varepsilon \nabla W V + \varepsilon \nabla V W + \varepsilon \frac{1}{2} \nabla |V|^2 = 0, \\ \eta|_{t=0} = \eta_0, V|_{t=0} = V_0. \end{cases} \quad (1.10)$$

Also because of (1.9) we get that  $\operatorname{curl} V = 0$  at any time meaning that for all  $l, k \in \overline{1, n}$  we have:

$$\partial_l V^k = \partial_k V^l. \quad (1.11)$$

The advantage of working with system (1.10) instead of (1.1) is two-folded. On one side certain commutators involving  $V$  behave better if its curl is 0 and on the other side, for some values of the  $a, b, c, d$  parameters e.g.  $a = c = d = 0, b > 0$  we need less regularity on the initial data  $V_0$ .

Let us establish some notations. The spaces  $L^p(\mathbb{R}^n)$  with  $p \in [1, \infty]$  will denote the classical Lebesgue spaces. Given  $s \in \mathbb{R}$  we will consider the following set of indices:

$$\begin{cases} s_1 = s + \operatorname{sgn}(b) - \operatorname{sgn}(c), \\ s_2 = s + \operatorname{sgn}(d) - \operatorname{sgn}(a), \\ s_3 = s + 1 - \operatorname{sgn}(a), \end{cases} \quad (1.12)$$

where the sign function  $\operatorname{sgn}$  is given by:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases}$$

and  $a, b, c, d$  are chosen as in (1.4). We will denote by  $H^s(\mathbb{R}^n)$  the classical Sobolev space of regularity index  $s$  with the norm

$$\|\eta\|_{H^s}^2 = \int_{\mathbb{R}^n} \left( 1 + |\xi|^2 \right)^s |\hat{\eta}(\xi)|^2 d\xi. \quad (1.13)$$

For any vector, matrix or 3-tensor field with components in  $H^s(\mathbb{R}^n)$  we denote its Sobolev norm by the square root of the sum of the squares of the Sobolev norms of its components. For any pair  $(\eta, V) \in H^{s_1}(\mathbb{R}^n) \times (H^{s_2}(\mathbb{R}^n))^n$  we will use the notation

$$\begin{aligned} \|(\eta, V)\|_s^2 &= \|\eta\|_{H^s}^2 + \varepsilon(b-c) \|\nabla \eta\|_{H^s}^2 + \varepsilon^2(-c)b \|\nabla^2 \eta\|_{H^s}^2 + \\ &+ \|V\|_{H^s}^2 + \varepsilon(d-a) \|\nabla V\|_{H^s}^2 + \varepsilon^2(-a)d \|\nabla^2 V\|_{H^s}^2, \end{aligned}$$

where  $\nabla^2 \eta = (\partial_{ij}^2 \eta)_{i,j}$  and  $\nabla^2 V = (\partial_{ij}^2 V^k)_{i,j,k}$ . Clearly,  $H^{s_1}(\mathbb{R}^n) \times (H^{s_2}(\mathbb{R}^n))^n$  endowed with  $\|(\cdot, \cdot)\|_s$  is a Banach space which is continuously imbedded in  $L^2(\mathbb{R}^n) \times (L^2(\mathbb{R}^n))^n$ .

Our approach is based on an energy method applied to spectrally localized equations. We first derive a priori estimates and we establish local existence and uniqueness of solutions for the general  $abcd$  system. Before we state the main result let us formalize the notion of long time existence of solutions for (1.10) in the next definitions.

**Definition 1.1.** *Let  $T > 0$  a positive real number. We will say that  $T$  is bounded from below by a  $O(\frac{1}{\varepsilon})$ -order quantity if there exists another positive real number  $C$ , independent of  $\varepsilon$  such that:*

$$T \geq \frac{C}{\varepsilon}.$$

Let us consider a Banach space  $(X, \|\cdot\|_X)$  which is continuously imbedded in  $L^2(\mathbb{R}^n) \times (L^2(\mathbb{R}^n))^n$ .

**Definition 1.2.** *Let us consider  $W \in (H^1(\mathbb{R}^n))^n$ . We say that we can establish long time existence and uniqueness of solutions for the equation (1.10) in  $X$  if for any  $(\eta_0, V_0) \in X$  there exists a positive time  $T > 0$ , an unique solution<sup>1</sup>  $(\eta, V) \in \mathcal{C}([0, T], X)$  of (1.10) and a function  $F : (0, +\infty) \rightarrow (0, +\infty)$  independent of  $\varepsilon$  such that:*

$$T \geq \frac{F(\|(\eta_0, V_0)\|_X)}{\varepsilon}.$$

**Remark 1.1.** *Of course, the function  $F$  appearing in the preceding definition is allowed to depend on  $a, b, c, d, W$  and on the dimension  $n$ .*

We are now in the position of stating our long time existence result:

**Theorem 1.** *Let  $a, b, c, d$  as in (1.4) excluding the two cases (1.8),  $b + d > 0$ . Let us consider  $s$  such that  $s > \frac{n}{2} + 1$  with  $n \geq 1$ . Let us also consider  $s_1, s_2$  and  $s_3$  defined by (1.12) and  $W \in (H^{s_3})^n$ . Then, we can establish long time existence and uniqueness of solutions for the equation (1.10) in  $H^{s_1} \times (H^{s_2})^n$ . Moreover, if we denote by  $T(\eta_0, V_0)$ , the maximal time of existence then there exists some  $T \in [0, T(\eta_0, V_0))$  which is bounded from below by an  $O(\frac{1}{\varepsilon})$ -order quantity and a function  $G : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $t \in [0, T]$  we have:*

$$\|(\eta, V)\|_s \leq G(\|(\eta_0, V_0)\|_s),$$

where  $G$  may depend on  $a, b, c, d, s, n$  but not on  $\varepsilon$ .

Theorem 1 is the consequence of a more general result that we obtain later in this paper. In fact, our method enables us (without any extra effort) to establish long time existence and uniqueness of solutions in the more general Besov spaces, thus achieving the critical regularity  $s = \frac{n}{2} + 1$ , see Theorem 3.

The rest of the paper is organized as follows. In Section 2 we establish all the basic energy estimates that we will need in order to prove Theorem 1. In Section 3 we prove that (1.10) admits an unique

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<sup>1</sup>The solution should be understood in the tempered distribution sense. See Definition 3.1.

solution and we establish an explosion criterion. The method used to construct the solution assures an existence time that is bounded below by a quantity of order  $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ . Finally in Section 4 we prove Theorem 1 namely we show that the solution of (1.10) constructed in Section 3 persists on a time of order  $O\left(\frac{1}{\varepsilon}\right)$ . The proof will be a by-product of some refined energy estimates that we prove in Section 2 and the explosion criteria established in Section 3. In Section 5.1 we discuss about the cases (1.8). We end the paper with Section 5.2 where we discuss the possibility of applying our method to the  $abcd$  systems in the case of a general bottom topography derived in [9], Section 2.

### 1.3 Notations

Because our proof makes use of elementary tensor calculus let us introduce some basic notations. For any vector field  $U : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we denote by  $\nabla U : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  and by  $\nabla^t U : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  the  $n \times n$  matrices defined by:

$$\begin{aligned} (\nabla U)_{ij} &= \partial_i U^j, \\ (\nabla^t U)_{ij} &= \partial_j U^i. \end{aligned}$$

In the same manner we define  $\nabla^2 U : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$  as:

$$(\nabla^2 U)_{ijk} = \partial_{ij}^2 U^k.$$

We will suppose that all vectors appearing are column vectors and thus the (classical) product between a matrix field  $A$  and a vector field  $U$  will be the vector<sup>2</sup>:

$$(AU)^i = A_{ij} U^j.$$

We will often write the contraction operation between  $\nabla^2 U$  and a vector field  $V$  by

$$(\nabla^2 U : V)_{ij} = \partial_{ij}^2 U^k V^k$$

If  $U, V : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are two vector fields and  $A, B : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  two matrix fields we denote:

$$\begin{aligned} UV &= U^i V^i, \quad A : B = A_{ij} B_{ij}, \\ \langle U, V \rangle_{L^2} &= \int U^i V^i, \quad \langle A, B \rangle_{L^2} = \int A_{ij} B_{ij} \\ \|U\|_{L^2}^2 &= \langle U, U \rangle_{L^2}, \quad \|A\|_{L^2}^2 = \langle A, A \rangle_{L^2} \\ \|\nabla^2 U\|_{L^2}^2 &= \int \nabla U : \nabla U = \int (\partial_{ij} U^k)^2 \end{aligned}$$

Also, the tensorial product of two vector fields  $U, V$  is defined as the matrix field  $U \otimes V : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  given by:

$$(U \otimes V)_{ij} = U^i V^j.$$

We have the following derivation rule: if  $u$  is a scalar field,  $U, V$  are vector fields and  $A : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  then:

$$\begin{aligned} \nabla \operatorname{div}(uU) &= \nabla^2 u U + \nabla u \operatorname{div} U + \nabla U \nabla u + u \nabla \operatorname{div} U \\ \nabla(UV) &= \nabla U V + \nabla V U \end{aligned}$$

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<sup>2</sup>From now on we will use the Einstein summation convention over repeated indices.

$$\nabla(AV) = \nabla^2 A : V + \nabla V A^t$$

If we suppose that  $\operatorname{curl} V = 0$  then the following integration by parts identity holds true:

$$\begin{aligned} \langle \nabla V : U, \nabla V \rangle_{L^2} &= \int (\nabla V : U) : \nabla V = \int \partial_{ij}^2 V^k U^k \partial_i V^j = \int \partial_{ik}^2 V^j U^k \partial_i V^j \\ &= -\frac{1}{2} \int \partial_k U^k (\partial_i V^j)^2 = -\frac{1}{2} \int \operatorname{div} U (\nabla V : \nabla V). \end{aligned} \quad (1.14)$$

Let  $\mathcal{C}$  be the annulus  $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$ . Let us choose two radial functions  $\chi \in \mathcal{D}(B(0, 4/3))$  and  $\varphi \in \mathcal{D}(\mathcal{C})$  valued in the interval  $[0, 1]$  and such that:

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1.$$

Let us denote by  $h = \mathcal{F}^{-1} \varphi$  and  $\tilde{h} = \mathcal{F}^{-1} \chi$ . For all  $u \in \mathcal{S}'$ , the nonhomogeneous dyadic blocks are defined as follows:

$$\begin{aligned} \Delta_j u &= 0 \quad \text{if } j \leq -2, \\ \Delta_{-1} u &= \chi(D) u = \tilde{h} \star u, \\ \Delta_j u &= \varphi(2^{-j} D) u = 2^{jd} \int_{\mathbb{R}^n} h(2^j y) u(x - y) dy \quad \text{if } j \geq 0. \end{aligned} \quad (1.15)$$

The high frequency cut-off operator  $S_j$  is defined by

$$S_j u = \sum_{k \leq j-1} \Delta_k u.$$

Let us define now the nonhomogeneous Besov spaces.

**Definition 1.3.** Let  $s \in \mathbb{R}$ ,  $r \in [1, \infty]$ . The Besov space  $B_{2,r}^s$  is the set of tempered distributions  $u \in \mathcal{S}'$  such that:

$$\|u\|_{B_{2,r}^s} := \left\| \left( 2^{js} \|\Delta_j u\|_{L^2} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Let us mention that  $H^s(\mathbb{R}^n) = B_{2,2}^s(\mathbb{R}^n)$  and that we have the following continuous embedding

$$B_{2,1}^s \hookrightarrow H^s \hookrightarrow B_{2,\infty}^s \hookrightarrow H^{s'},$$

for all  $s' < s$ . Some basic properties about Besov spaces can be found in the Appendix. For more details and full proofs we refer to [3].

Let us consider  $\varepsilon \leq 1$  and  $s > 0$ ,  $r \in [1, \infty]$ . For all  $(\eta, V) \in B_{2,r}^{s_1}(\mathbb{R}^n) \times (B_{2,r}^{s_2}(\mathbb{R}^n))^n$  we introduce the following quantities:

$$\begin{aligned} U_j^2 &= U_j^2(\eta, V) = \int \eta_j^2 + \varepsilon(b-c) |\nabla \eta_j|^2 + \varepsilon^2(-c)b (\nabla^2 \eta_j : \nabla^2 \eta_j) \\ &\quad + \int V_j^2 + \varepsilon(d-a) (\nabla V_j : \nabla V_j) + \varepsilon^2(-a)d (\nabla^2 V_j : \nabla^2 V_j), \end{aligned} \quad (1.16)$$

and

$$\begin{aligned}
 U_s^2 = U_s^2(\eta, V) &= \|\eta\|_{B_{2,r}^s}^2 + \varepsilon(b-c)\|\nabla\eta\|_{B_{2,r}^s}^2 + \varepsilon^2(-c)b\|\nabla^2\eta\|_{B_{2,r}^s}^2 \\
 &+ \|V\|_{B_{2,r}^s}^2 + \varepsilon(d-a)\|\nabla V\|_{B_{2,r}^s}^2 + \varepsilon^2(-a)d\|\nabla^2 V\|_{B_{2,r}^s}^2.
 \end{aligned} \tag{1.17}$$

where  $(\eta_j, V_j) := (\Delta_j \eta, \Delta_j V)$  are the frequency-localized dyadic blocks defined by relation (1.15). It is easy to check that  $U_s(\eta, V)$  is a norm on the space  $B_{2,r}^{s_1}(\mathbb{R}^n) \times (B_{2,r}^{s_2}(\mathbb{R}^n))^n$ . Also, it transpires that:

$$\left\| (2^{js} U_j)_{j \in \mathbb{Z}} \right\|_{\ell^r} \approx U_s.$$

Some times we will also use the notation

$$\|(\eta, V)\|_s = U_s(\eta, V),$$

for all  $(\eta, V) \in B_{2,r}^{s_1}(\mathbb{R}^n) \times (B_{2,r}^{s_2}(\mathbb{R}^n))^n$ . We observe that  $(B_{2,r}^{s_1}(\mathbb{R}^n) \times (B_{2,r}^{s_2}(\mathbb{R}^n))^n, \|(\cdot, \cdot)\|_s)$  is a Banach space. In order to ease the notations we will rather write  $B_{2,r}^s$  instead of  $B_{2,r}^{s_1}(\mathbb{R}^n)$ . Another quantity that will play an important role in the following analysis is:

$$\begin{aligned}
 N_j^2 = N_j^2(\eta, V) &= \int (1 + \varepsilon \|\eta\|_{L^\infty}) \eta_j^2 + \varepsilon(b-c)(1 + \varepsilon \|\eta\|_{L^\infty}) |\nabla \eta_j|^2 \\
 &+ \int \varepsilon^2(-c)b(1 + \varepsilon \|\eta\|_{L^\infty}) (\nabla^2 \eta_j : \nabla^2 \eta_j) \\
 &+ \int (1 + \varepsilon \eta + \varepsilon \|\eta\|_{L^\infty}) V_j^2 + \varepsilon(d-a+d\varepsilon\eta+d\varepsilon\|\eta\|_{L^\infty}) (\nabla V_j : \nabla V_j) \\
 &+ \int \varepsilon^2(-a)d(1 + \varepsilon \|\eta\|_{L^\infty}) (\nabla^2 V_j : \nabla^2 V_j),
 \end{aligned} \tag{1.18}$$

which satisfies:

$$U_j(\eta, V) \lesssim N_j(\eta, V) \lesssim (1 + 2\varepsilon \|\eta\|_{L^\infty})^{\frac{1}{2}} U_j(\eta, V) \tag{1.19}$$

Denoting by

$$N_s = N_s(\eta, V) = \left\| (2^{js} N_j(\eta, V))_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} \tag{1.20}$$

we obviously have that

$$U_s(\eta, V) \lesssim N_s(\eta, V) \lesssim (1 + 2\varepsilon \|\eta\|_{L^\infty})^{\frac{1}{2}} U_s(\eta, V). \tag{1.21}$$

## 2 Energy-type identities

We begin by localizing equation (1.10) in Fourier space thus obtaining that:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta_j + \operatorname{div} V_j + a\varepsilon \operatorname{div} \Delta V_j + \varepsilon W \nabla \eta_j + \varepsilon V \nabla \eta_j + \varepsilon \eta \operatorname{div} V_j = \varepsilon R_{1j} \\ (I - \varepsilon d \Delta) \partial_t V_j + \nabla \eta_j + c\varepsilon \nabla \Delta \eta_j + \varepsilon \nabla V_j W + \varepsilon \nabla V_j V = \varepsilon R_{2j} \\ \eta_j|_{t=0} = \Delta_j \eta_0, V_j|_{t=0} = \Delta_j V_0 \end{cases} \tag{2.1}$$

where the remainder terms are given by<sup>3</sup>:

$$\begin{cases} R_{1j} = [W, \Delta_j] \nabla \eta + [V, \Delta_j] \nabla \eta + [\eta, \Delta_j] \operatorname{div} V \\ R_{2j} = [W, \Delta_j] \nabla V + [V, \Delta_j] \nabla V - \frac{1}{2} \nabla \Delta_j (|W|^2) - \Delta_j (\nabla W V). \end{cases} \tag{2.2}$$

<sup>3</sup>From now on, we agree that if  $A, B$  are two operator then the operator  $[A, B]$  is given by:

$$[A, B] = AB - BA.$$



Let us establish our first useful identity. We multiply the first equation in (2.1) by  $\eta_j$  and the second one with  $V_j$  and by adding them up and integrating in space we get that<sup>4</sup>:

$$\begin{aligned} \frac{1}{2} \partial_t \left( \|\eta_j\|_{L^2}^2 + \varepsilon b \|\nabla \eta_j\|_{L^2}^2 + \|V_j\|_{L^2}^2 + \varepsilon d \|\nabla V_j\|_{L^2}^2 \right) + a\varepsilon \int \eta_j \operatorname{div} \Delta V_j + c\varepsilon \int V_j \nabla \Delta \eta_j \\ + \varepsilon \int \eta_j \operatorname{div} V_j = \frac{\varepsilon}{2} \int \operatorname{div} V (\eta_j^2 + V_j^2) + \varepsilon \int R_{1j} \eta_j + \varepsilon \int R_{2j} V_j. \end{aligned} \quad (2.3)$$

Let us denote by  $T_1$  the right hand side of the above identity:

$$T_1 = \frac{\varepsilon}{2} \int \operatorname{div} V (\eta_j^2 + V_j^2) + \varepsilon \int R_{1j} \eta_j + \varepsilon \int R_{2j} V_j. \quad (2.4)$$

Next, we wish to derive similar identities involving the quantities  $\nabla \eta_j, \nabla^2 \eta_j$  and  $\nabla V_j, \nabla^2 V_j$ . In order to do so, let us observe that applying  $\nabla$  to the first equation in (1.10) gives us:

$$(I - \varepsilon b \Delta) \partial_t \nabla \eta + \nabla \operatorname{div} V + a\varepsilon \nabla \operatorname{div} \Delta V + \varepsilon \nabla W \nabla \eta + \varepsilon \nabla^2 \eta W + \varepsilon \nabla \operatorname{div} (\eta V) = 0$$

and that by applying  $\Delta_j$  we end up with:

$$\begin{aligned} (I - \varepsilon b \Delta) \partial_t \nabla \eta_j + \nabla \operatorname{div} V_j + a\varepsilon \nabla \operatorname{div} \Delta V_j + \varepsilon \nabla^2 \eta_j W + \varepsilon \nabla^2 \eta_j V + \varepsilon \eta_j \nabla \operatorname{div} V_j \\ + \varepsilon \operatorname{div} V_j \nabla \eta + \varepsilon \nabla V_j \nabla \eta = -\varepsilon \Delta_j (\nabla W \nabla \eta) + \varepsilon R_{3j}, \end{aligned} \quad (2.5)$$

where

$$R_{3j} = [W, \Delta_j] \nabla^2 \eta + [V, \Delta_j] \nabla^2 \eta + [\eta, \Delta_j] \nabla \operatorname{div} V + [\nabla \eta, \Delta_j] \operatorname{div} V + [\nabla \eta, \Delta_j] \nabla V.$$

We multiply (2.5) with  $-c\varepsilon \nabla \eta_j$  and by integration we get<sup>5</sup>:

$$\begin{aligned} \frac{-c\varepsilon}{2} \partial_t \left( \int (|\nabla \eta_j|^2 + \varepsilon b \nabla^2 \eta_j : \nabla^2 \eta_j) \right) - c\varepsilon \int \nabla \operatorname{div} V_j \nabla \eta_j - ac\varepsilon^2 \int \nabla \operatorname{div} \Delta V_j \nabla \eta_j \\ - c\varepsilon^2 \int \eta_j \nabla \operatorname{div} V_j \nabla \eta_j = T_2 \end{aligned} \quad (2.6)$$

with  $T_2$  given by

$$\begin{aligned} T_2 = c\varepsilon^2 \int (\nabla^2 \eta_j W) \nabla \eta_j + c\varepsilon^2 \int (\nabla^2 \eta_j V) \nabla \eta_j \\ + c\varepsilon^2 \int \operatorname{div} V_j \nabla \eta_j \nabla \eta_j + c\varepsilon^2 \int (\nabla V_j \nabla \eta) \nabla \eta_j + c\varepsilon^2 \int \Delta_j (\nabla W \nabla \eta) \nabla \eta_j - c\varepsilon^2 \int R_{3j} \nabla \eta_j. \end{aligned} \quad (2.7)$$

We proceed similarly with the second equation in (1.10) and we obtain:

$$(I - \varepsilon d \Delta) \partial_t \nabla V + \nabla^2 \eta + \varepsilon c \nabla^2 \Delta \eta + \varepsilon \frac{1}{2} \nabla^2 |W|^2 + \varepsilon (\nabla^2 W : V + \nabla V \nabla^t W)$$

---

<sup>4</sup>Observe that here we use the fact that  $\operatorname{curl} V = 0$ . Indeed, under (1.11) we have

$$\int \nabla V_j V_j = \frac{1}{2} \int V \nabla |V_j|^2 = -\frac{1}{2} \int \operatorname{div} V |V_j|^2.$$

<sup>5</sup>Here, we use the fact that  $\operatorname{div} W = 0$ . We get that:

$$\varepsilon \int W \nabla \eta_j \eta_j = -\frac{\varepsilon}{2} \int \operatorname{div} W \eta_j^2 = 0.$$

$$+\varepsilon (\nabla^2 V : W + \nabla W \nabla^t V) + \varepsilon (\nabla^2 V : V + \nabla V \nabla^t V) = 0.$$

We localize the last equation and we get that:

$$\begin{aligned} & (I - \varepsilon d \Delta) \partial_t \nabla V_j + \nabla^2 \eta_j + \varepsilon c \nabla^2 \Delta \eta_j + \varepsilon \frac{1}{2} \Delta_j \nabla^2 |W|^2 + \varepsilon \Delta_j (\nabla^2 W : V) \\ & + \varepsilon \Delta_j (\nabla V \nabla^t W) + \varepsilon \nabla^2 V_j : W + \varepsilon \Delta_j (\nabla W \nabla^t V) + \varepsilon \nabla^2 V_j : V + \varepsilon \nabla V_j \nabla^t V = \varepsilon R_{4j}, \end{aligned} \quad (2.8)$$

where:

$$R_{4j} = [W, \Delta_j] \nabla^2 V + [V, \Delta_j] \nabla^2 V + [\nabla^t V, \Delta_j] \nabla V.$$

We contract (2.8) with  $-a\varepsilon \nabla V_j$  and by integration we get that:

$$\frac{-a\varepsilon}{2} \partial_t \left( \int \nabla V_j : \nabla V_j + \varepsilon d \nabla^2 V_j : \nabla^2 V_j \right) - a\varepsilon \int \nabla^2 \eta_j : \nabla V_j - ac\varepsilon^2 \int \nabla^2 \Delta \eta_j : \nabla V_j = T_3 \quad (2.9)$$

with  $T_3$  given by:

$$\begin{aligned} T_3 = & a\varepsilon^2 \int (\nabla^2 V_j : W) : \nabla V_j + a\varepsilon^2 \int (\nabla^2 V_j : V) : \nabla V_j + a\varepsilon^2 \int \nabla V_j \nabla^t V : \nabla V_j - a\varepsilon^2 \int R_{4j} : \nabla V_j \\ & + a\varepsilon^2 \int \left( \frac{1}{2} \Delta_j \nabla^2 |W|^2 + \Delta_j (\nabla W \nabla^t V) + \Delta_j (\nabla^2 W : V) + \Delta_j (\nabla V \nabla^t W) \right) : \nabla V_j \end{aligned} \quad (2.10)$$

Let us add up identities (2.6) and (2.9) to get that<sup>6</sup>:

$$\begin{aligned} & \partial_t \left( \int \left\{ \frac{-c\varepsilon}{2} (|\nabla \eta_j|^2 + \varepsilon b \nabla^2 \eta_j : \nabla^2 \eta_j) + \frac{-a\varepsilon}{2} (\nabla V_j : \nabla V_j + \varepsilon d \nabla^2 V_j : \nabla^2 V_j) \right\} \right) \\ & - a\varepsilon \int \nabla^2 \eta_j : \nabla V_j - c\varepsilon \int \nabla \operatorname{div} V_j \nabla \eta_j - c\varepsilon^2 \int \eta \nabla \operatorname{div} V_j \nabla \eta_j = T_2 + T_3. \end{aligned}$$

Finally add up (2.3) to the last identity in order to obtain<sup>7</sup>:

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon (b - c) |\nabla \eta_j|^2 + \varepsilon^2 (-c) b (\nabla^2 \eta_j : \nabla^2 \eta_j) \right) \\ & + \frac{1}{2} \partial_t \left( \int |V_j|^2 + \varepsilon (d - a) (\nabla V_j : \nabla V_j) + \varepsilon^2 (-a) d (\nabla^2 V_j : \nabla^2 V_j) \right) \\ & + \varepsilon \int \eta \eta_j \operatorname{div} V_j - c\varepsilon^2 \int \eta \nabla \operatorname{div} V_j \nabla \eta_j = T_1 + T_2 + T_3. \end{aligned} \quad (2.11)$$

## 2.1 Refined energy-type identities

As we have already seen in (1.6), the  $abcd$  system possesses a formally conserved energy. The types of estimates that we establish in this section, resemble very much to this conserved energy and will be the equivalent of the ones obtained in [6] pages 617 and 625, with  $\Delta_j \eta$  and  $\Delta_j V$  instead of  $\eta$  and  $V$ . Of course this is the key ingredient for obtaining long time existence results that allow initial data to lie in larger spaces.

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<sup>6</sup>Observe that by integration by parts we get

$$-ac\varepsilon^2 \int \nabla \operatorname{div} \Delta V_j : \nabla \eta_j - ac\varepsilon^2 \int \nabla^2 \Delta \eta_j : \nabla V_j = 0.$$

<sup>7</sup>Observe that the terms  $a\varepsilon \int \eta_j \operatorname{div} \Delta V_j - a\varepsilon \int \nabla^2 \eta_j : \nabla V_j$  and  $c\varepsilon \int V_j \nabla \Delta \eta_j - c\varepsilon \int \nabla \operatorname{div} V_j : \nabla \eta_j$  are both 0.

Having established identity (2.11) we observe that the terms  $\varepsilon \int \eta \eta_j \operatorname{div} V_j$  and  $-c\varepsilon^2 \int \eta \nabla \operatorname{div} V_j \nabla \eta_j$  prevent us from directly applying a Gronwall type argument and establishing long time existence. In order to repair this inconvenience let us multiply the second equation of (2.1) with  $\varepsilon \eta V_j$ . We thus get:

$$\begin{aligned} & \langle \varepsilon (I - d\Delta) \partial_t V_j, \eta V_j \rangle_{L^2} + \varepsilon \int \eta \nabla \eta_j V_j + c\varepsilon^2 \int \eta \nabla \Delta \eta_j V_j \\ &= -\varepsilon^2 \int (\nabla V_j W) (\eta V_j) - \varepsilon^2 \int (\nabla V_j V) (\eta V_j) + \varepsilon^2 \int \eta R_{2j} V_j. \end{aligned}$$

Observing that the first term writes:

$$\begin{aligned} \langle \varepsilon (I - d\Delta) \partial_t V_j, \eta V_j \rangle_{L^2} &= \frac{1}{2} \partial_t \left( \int \varepsilon \eta |V_j|^2 + \varepsilon^2 d \eta \nabla V_j : \nabla V_j \right) - \frac{\varepsilon}{2} \int \partial_t \eta |V_j|^2 \\ &\quad - \varepsilon^2 \frac{d}{2} \int \partial_t \eta \nabla V_j : \nabla V_j + \varepsilon^2 d \langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \rangle_{L^2} \end{aligned}$$

we write that

$$\frac{1}{2} \partial_t \left( \int \varepsilon \eta |V_j|^2 + \varepsilon^2 d \eta \nabla V_j : \nabla V_j \right) + \varepsilon \int \eta \nabla \eta_j V_j + c\varepsilon^2 \int \eta \nabla \Delta \eta_j V_j = T_4 \quad (2.12)$$

with  $T_4$  given by:

$$\begin{aligned} T_4 &= -\varepsilon^2 \int (\nabla V_j W) \eta V_j - \varepsilon^2 \int (\nabla V_j V) \eta V_j + \varepsilon^2 \int \eta R_{2j} V_j + \\ &\quad + \frac{\varepsilon}{2} \int \partial_t \eta |V_j|^2 + \varepsilon^2 \frac{d}{2} \int \partial_t \eta \nabla V_j : \nabla V_j - \varepsilon^2 d \langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \rangle_{L^2}. \end{aligned} \quad (2.13)$$

We add up (2.11) and (2.12) in order to obtain:

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon (b - c) |\nabla \eta_j|^2 + \varepsilon^2 (-c) b (\nabla^2 \eta_j : \nabla^2 \eta_j) \right) \\ &+ \frac{1}{2} \partial_t \left( \int (1 + \varepsilon \eta) |V_j|^2 + \varepsilon (d - a + d\varepsilon \eta) (\nabla V_j : \nabla V_j) + \varepsilon^2 (-a) d (\nabla^2 V_j : \nabla^2 V_j) \right) \\ &= T_0 + T_1 + T_2 + T_3 + T_4, \end{aligned} \quad (2.14)$$

with

$$T_0 = \varepsilon \int \nabla \eta \eta_j V_j - c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j + c\varepsilon^2 \int \nabla \eta \nabla \eta_j \operatorname{div} V_j.$$

## 2.2 Estimates for the $T_i$ 's

Having established the energy identity (2.14), we proceed by conveniently bounding the RHS term. We want to obtain a bound of the form:

$$T_0 + T_1 + T_2 + T_3 + T_4 \leq \varepsilon P(U_j, U_s)$$

with  $P(x, y)$  some polynomial function with coefficients not depending on  $\varepsilon$ . We suppose that

$$s > \frac{n}{2} + 1 \text{ or } s = \frac{n}{2} + 1 \text{ and } r = 1.$$

Moreover,  $C > 0$  will denote a generic positive constant depending only on the dimension  $n$  and on  $s$ . All the estimates established here are valid for the  $a, b, c, d$  parameters satisfying (1.4) with  $b + d > 0$ . The case:

$$a = d = 0, \quad b > 0, \quad c < 0 \quad (2.15)$$

needs more attention and we will investigate it in Section 2.4, thus for the moment we do all the computation supposing that case (2.15) does not occur.

We claim that we can bound  $T_0 + T_1 + T_2 + T_3 + T_4$  in such a way that the following estimate holds true:

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon (b - c) |\nabla \eta_j|^2 + \varepsilon^2 (-c) b (\nabla^2 \eta_j : \nabla^2 \eta_j) \right) \\ & + \frac{1}{2} \partial_t \left( \int (1 + \varepsilon \eta) V_j^2 + \varepsilon (d - a + d\varepsilon \eta) (\nabla V_j : \nabla V_j) + \varepsilon^2 (-a) d (\nabla^2 V_j : \nabla^2 V_j) \right) \\ & \leq \varepsilon C U_j \{ U_j (U_s + H U_s + U_s^2) + c_j(t) (1 + U_s) (H^2 + H U_s + U_s^2) \}, \end{aligned} \quad (2.16)$$

where  $C > 0$  depends only on  $a, b, c, d, n$  and  $s$  but not on  $\varepsilon$ ,  $(c_j(t))_j$  is a sequence with  $(2^{js} c_j(t))_j \in \ell^r(\mathbb{Z})$ , having norm 1 and finally  $H$  is defined by:

$$H = \|W\|_{B_{2,r}^s} + \|\nabla W\|_{B_{2,r}^s} - \operatorname{sgn}(a) \sqrt{\varepsilon} \|\nabla^2 W\|_{B_{2,r}^s}.$$

Let us detail this below. Regarding  $T_0$ , we begin by observing that:

$$\varepsilon \int \nabla \eta \eta_j V_j \leq \varepsilon \|\nabla \eta\|_{L^\infty} \|\eta_j\|_{L^2} \|V_j\|_{L^2} \leq \varepsilon U_j^2 U_s.$$

Next, if  $a = d = 0$  then  $b > 0$  and we get that<sup>8</sup>:

$$\begin{aligned} & -c\varepsilon^2 \int \Delta \eta \nabla \eta_j V_j + c\varepsilon^2 \int \nabla \eta \nabla \eta_j \operatorname{div} V_j \\ & = -c\varepsilon^2 \int \Delta \eta \nabla \eta_j V_j - c\varepsilon^2 \int \nabla^2 \eta \nabla \eta_j V_j - c\varepsilon^2 \int \nabla^2 \eta_j \nabla \eta V_j \\ & \leq -c\varepsilon^2 (\|\Delta \eta\|_{L^\infty} \|\nabla \eta_j\|_{L^2} \|V_j\|_{L^2} + \|\nabla^2 \eta\|_{L^\infty} \|\nabla \eta_j\|_{L^2} \|V_j\|_{L^2} \\ & \quad + \|\nabla \eta\|_{L^\infty} \|\nabla^2 \eta_j\|_{L^2} \|V_j\|_{L^2}) \\ & \leq \varepsilon C \max \left( \frac{-c}{b-c}, \sqrt{\frac{-c}{b}} \right) U_j^2 U_s. \end{aligned}$$

If at least one of  $a, d$  is not 0 then we write:

$$\begin{aligned} & -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j + c\varepsilon^2 \int \nabla \eta \nabla \eta_j \operatorname{div} V_j \\ & = c\varepsilon^2 \int (\nabla^2 \eta V_j) \nabla \eta_j + c\varepsilon^2 \int (\nabla V_j \nabla \eta) \nabla \eta_j + c\varepsilon^2 \int \nabla \eta \nabla \eta_j \operatorname{div} V_j \\ & \leq -c\varepsilon^2 (\|\nabla^2 \eta\|_{L^\infty} \|V_j\|_{L^2} \|\nabla \eta_j\|_{L^2} + \|\nabla \eta\|_{L^\infty} \|\nabla V_j\|_{L^2} \|\nabla \eta_j\|_{L^2} \\ & \quad + \|\nabla \eta\|_{L^\infty} \|\nabla \eta_j\|_{L^2} \|\operatorname{div} V_j\|_{L^2}) \\ & \leq \varepsilon C \max \left( \frac{-c}{b-c}, \frac{-c}{d-a} \right) U_j^2 U_s. \end{aligned}$$

We choose

$$C_{abcd}^1 = \begin{cases} \max \left( \frac{-c}{b-c}, \sqrt{\frac{-c}{b}} \right) & \text{if } d - a = 0, \\ \max \left( \frac{-c}{b-c}, \frac{-c}{d-a} \right) & \text{if } d - a > 0. \end{cases} \quad (2.17)$$

<sup>8</sup>From now on, we adopt the convention  $\frac{0}{0} = 0$ .

Consequently, we have:

$$T_0 \leq \varepsilon C C_{abcd}^1 U_j^2 U_s. \quad (2.18)$$

Next, let us analyze  $T_1$ . According to Proposition 6.7 and Proposition 6.2 we get that:

$$\begin{aligned} \|R_{1j}\|_{L^2} &\leq C c_j^1(t) \left( \|\nabla W\|_{B_{2,r}^{s-1}} \|\eta\|_{B_{2,r}^s} + \|\nabla V\|_{B_{2,r}^{s-1}} \|\eta\|_{B_{2,r}^s} + \|\nabla \eta\|_{B_{2,r}^{s-1}} \|V\|_{B_{2,r}^s} \right), \\ \|R_{2j}\|_{L^2} &\leq C c_j^2(t) \left( \left( \|\nabla W\|_{B_{2,r}^{s-1}} + \|\nabla V\|_{B_{2,r}^{s-1}} \right) \|V\|_{B_{2,r}^s} + \left( \|W\|_{B_{2,r}^s} + \|V\|_{B_{2,r}^s} \right) \|\nabla W\|_{B_{2,r}^s} \right), \end{aligned}$$

with  $(2^{js} c_j^i(t)) \in \ell^r(\mathbb{Z})$ ,  $i = 1, 2$ , with norm 1. In order to avoid mentioning each time, from now on, for all natural number,  $i \in \mathbb{N}$ ,  $(c_j^i(t))_{j \in \mathbb{Z}}$  will be a sequence such that  $(2^{js} c_j^i(t))_{j \in \mathbb{Z}} \in \ell^r(\mathbb{Z})$  with norm 1. Recall that  $H$  stands for the following quantity:

$$H = \|W\|_{B_{2,r}^s} + \|\nabla W\|_{B_{2,r}^s} - \operatorname{sgn}(a) \sqrt{\varepsilon} \|\nabla^2 W\|_{B_{2,r}^s} \quad (2.19)$$

and rewrite the previous inequalities (of course the constant  $C = C(n, s)$  changes whenever necessary) as:

$$\|R_{1j}\|_{L^2} \leq C c_j^1(t) U_s (H + U_s), \quad (2.20)$$

$$\|R_{2j}\|_{L^2} \leq C c_j^2(t) (H^2 + U_s (U_s + H)). \quad (2.21)$$

We get that

$$\begin{aligned} T_1 &= \frac{\varepsilon}{2} \int \operatorname{div} V (\eta_j^2 + V_j^2) + \varepsilon \int R_{1j} \eta_j + \varepsilon \int R_{2j} V_j \\ &\leq \varepsilon \|\operatorname{div} V\|_{L^\infty} \left( \|\eta_j\|_{L^2}^2 + \|V_j\|_{L^2}^2 \right) + \varepsilon C c_j^3(t) U_j (H^2 + U_s (U_s + H)) \\ &\leq C \varepsilon (U_j^2 U_s + c_j^3(t) U_j (H^2 + H U_s + U_s^2)). \end{aligned} \quad (2.22)$$

We turn our attention towards  $T_2$ :

$$\begin{aligned} T_2 &= c\varepsilon^2 \int (\nabla^2 \eta_j W) \nabla \eta_j + c\varepsilon^2 \int (\nabla^2 \eta_j V) \nabla \eta_j + c\varepsilon^2 \int \operatorname{div} V_j \nabla \eta \nabla \eta_j + c\varepsilon^2 \int \nabla V_j \nabla \eta \nabla \eta_j \\ &\quad + c\varepsilon^2 \int \Delta_j (\nabla W \nabla \eta) \nabla \eta_j - c\varepsilon^2 \int R_{3j} \nabla \eta_j. \end{aligned}$$

According to Proposition 6.2 and Proposition 6.7 we get that

$$\begin{aligned} &\|\Delta_j (\nabla W \nabla \eta)\|_{L^2} + \|R_{3j}\|_{L^2} \\ &\leq C c_j^4(t) \left( \|\nabla W\|_{B_{2,r}^s} \|\nabla \eta\|_{B_{2,r}^s} + \|\nabla W\|_{B_{2,r}^{s-1}} \|\nabla \eta\|_{B_{2,r}^s} \right. \\ &\quad \left. + \|\nabla V\|_{B_{2,r}^{s-1}} \|\nabla \eta\|_{B_{2,r}^s} + \|\nabla \eta\|_{B_{2,r}^{s-1}} \|\nabla V\|_{B_{2,r}^s} + \|\nabla^2 \eta\|_{B_{2,r}^{s-1}} \|V\|_{B_{2,r}^s} \right) \\ &\leq C c_j^4(t) \left( \|\nabla W\|_{B_{2,r}^s} \|\nabla \eta\|_{B_{2,r}^s} + \|W\|_{B_{2,r}^s} \|\nabla \eta\|_{B_{2,r}^s} \right. \\ &\quad \left. + \|\eta\|_{B_{2,r}^s} \|\nabla V\|_{B_{2,r}^s} + \|\nabla \eta\|_{B_{2,r}^s} \|V\|_{B_{2,r}^s} \right). \end{aligned} \quad (2.23)$$

We observe that it is here that we need the restriction (2.15) on the parameters. Indeed if  $a = d = 0$  and  $c < 0$ , then we only have  $V \in B_{2,r}^s$  hence  $U_s$  cannot control  $\|\nabla V\|_{B_{2,r}^s}$ . Let us observe that:

$$\int (\nabla^2 \eta_j W) \nabla \eta_j = -\frac{1}{2} \int |\nabla \eta_j|^2 \operatorname{div} W = 0$$

and

$$\int (\nabla^2 \eta_j V) \nabla \eta_j = -\frac{1}{2} \int |\nabla \eta_j|^2 \operatorname{div} V.$$

We infer that:

$$\begin{aligned} T_2 &= c\varepsilon^2 \int \operatorname{div} V_j \nabla \eta \nabla \eta_j + c\varepsilon^2 \int (\nabla V_j \nabla \eta) \nabla \eta_j + c\varepsilon^2 \int \Delta_j (\nabla W \nabla \eta) \nabla \eta_j - c\varepsilon^2 \int R_{3j} \nabla \eta_j \\ &\leq \frac{-c}{2} \varepsilon^2 \left\{ \|\operatorname{div} V\|_{L^\infty} \|\nabla \eta_j\|_{L^2}^2 + 2 \|\nabla \eta\|_{L^\infty} \|\nabla \eta_j\|_{L^2} \|\nabla V_j\|_{L^2} \right. \\ &\quad \left. + C c_j^4(t) \|\nabla \eta_j\|_{L^2} \left( \|\nabla W\|_{B_{2,r}^s} \|\nabla \eta\|_{B_{2,r}^s} + \|W\|_{B_{2,r}^s} \|\nabla \eta\|_{B_{2,r}^s} \right. \right. \\ &\quad \left. \left. + \|\eta\|_{B_{2,r}^s} \|\nabla V\|_{B_{2,r}^s} + \|\nabla \eta\|_{B_{2,r}^s} \|V\|_{B_{2,r}^s} \right) \right\} \\ &\leq \varepsilon C \max \left( \frac{-c}{b-c}, \frac{-c}{d-a} \right) (U_j^2 U_s + c_j^4(t) U_j H U_s). \end{aligned} \quad (2.24)$$

We let

$$C_{abcd}^2 = \max \left( \frac{-c}{b-c}, \frac{-c}{d-a} \right). \quad (2.25)$$

Let us estimate  $T_3$ . As above, owing to Proposition 6.7 we may bound  $R_{4j}$  in the following manner:

$$\begin{aligned} -a\varepsilon^2 \|R_{4j}\|_{L^2} &\leq -a\varepsilon^2 C c_j^5(t) \left( \|\nabla W\|_{B_{2,r}^{s-1}} \|\nabla V\|_{B_{2,r}^s} + \|\nabla V\|_{B_{2,r}^{s-1}} \|\nabla V\|_{B_{2,r}^s} + \|\nabla \nabla^t V\|_{B_{2,r}^{s-1}} \|V\|_{B_{2,r}^s} \right) \\ &\leq -a\varepsilon^2 C c_j^5(t) \left( \|W\|_{B_{2,r}^s} \|\nabla V\|_{B_{2,r}^s} + \|V\|_{B_{2,r}^s} \|\nabla V\|_{B_{2,r}^s} \right) \\ &\leq C \varepsilon^{\frac{3}{2}} \frac{-a}{\sqrt{d-a}} c_j^5(t) U_s (H + U_s). \end{aligned} \quad (2.26)$$

Also, we can write due to Proposition 6.2:

$$\begin{aligned} -a\varepsilon^2 \left( \frac{1}{2} \|\Delta_j \nabla^2 |W|^2\|_{L^2} + \|\Delta_j (\nabla W \nabla^t V)\|_{L^2} + \|\Delta_j (\nabla^2 W : V)\|_{L^2} + \|\Delta_j (\nabla V \nabla^t W)\|_{L^2} \right) \\ \leq -a\varepsilon^2 C c_j^6(t) \left( \|\nabla^2 W\|_{B_{2,r}^s} \|W\|_{B_{2,r}^s} + \|\nabla W\|_{B_{2,r}^s}^2 + \|\nabla W\|_{B_{2,r}^s} \|\nabla V\|_{B_{2,r}^s} \right. \\ \left. + \|\nabla^2 W\|_{B_{2,r}^s} \|V\|_{B_{2,r}^s} + \|\nabla W\|_{B_{2,r}^s} \|\nabla V\|_{B_{2,r}^s} \right) \\ \leq C \varepsilon^{\frac{3}{2}} \max \left( -a, \frac{-a}{\sqrt{d-a}} \right) c_j^6(t) (H^2 + H U_s). \end{aligned}$$

Then, using the integration by parts identity (1.14) we get that:

$$\begin{aligned} T_3 &= a\varepsilon^2 \int (\nabla^2 V_j : W) : \nabla V_j + a\varepsilon^2 \int (\nabla^2 V_j : V) : \nabla V_j \\ &\quad + a\varepsilon^2 \int \nabla V_j \nabla^t V : \nabla V_j - a\varepsilon^2 \int R_{4j} : \nabla V_j \\ &+ a\varepsilon^2 \int \left( \frac{1}{2} \Delta_j \nabla^2 |W|^2 + \Delta_j (\nabla W \nabla^t V) + \Delta_j (\nabla^2 W : V) + \Delta_j (\nabla V \nabla^t W) \right) : \nabla V_j \\ &\leq -a\varepsilon^2 \left( \frac{1}{2} \|\operatorname{div} V\|_{L^\infty} \|\nabla V_j\|_{L^2}^2 + \|\nabla^t V\|_{L^\infty} \|\nabla V_j\|_{L^2}^2 \right) + \\ &+ C \varepsilon \max \left( \frac{-a}{\sqrt{d-a}}, \frac{-a}{d-a} \right) \left( c_j^5(t) \varepsilon^{-\frac{1}{2}} U_j U_s (H + U_s) + c_j^6(t) \varepsilon^{-\frac{1}{2}} U_j (H^2 + H U_s) \right) \end{aligned}$$

$$\leq \varepsilon C \max \left( \frac{-a}{\sqrt{d-a}}, \frac{-a}{d-a} \right) (U_j^2 U_s + c_j^7(t) U_j (H^2 + H U_s + U_s^2)) \quad (2.27)$$

and as before, let us denote by

$$C_{abcd}^3 = \max \left( \frac{-a}{\sqrt{d-a}}, \frac{-a}{d-a} \right). \quad (2.28)$$

Finally let us turn our attention towards  $T_4$ . Let us write:

$$T_4 = G_4 + B_4,$$

with

$$B_4 = \frac{\varepsilon}{2} \int \partial_t \eta |V_j|^2 + \varepsilon^2 \frac{d}{2} \int \partial_t \eta \nabla V_j : \nabla V_j - \varepsilon^2 d \langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \rangle_{L^2}$$

and observe that by integration by parts and (2.21)

$$\begin{aligned} G_4 &= -\varepsilon^2 \int (\nabla V_j W) \eta V_j - \varepsilon^2 \int (\nabla V_j V) \eta V_j + \varepsilon^2 \int \eta R_{2j} V_j \\ &\leq \frac{\varepsilon^2}{2} \int \left( (\operatorname{div}(\eta W) + \operatorname{div}(\eta V)) |V_j|^2 \right) + C \varepsilon^2 c_j^2(t) \|V_j\|_{L^2} \|\eta\|_{L^\infty} (H^2 + H U_s + U_s^2) \\ &\leq C \varepsilon (U_j^2 (H U_s + U_s^2) + c_j^8(t) U_j U_s (H^2 + H U_s + U_s^2)). \end{aligned} \quad (2.29)$$

Let us now analyze the term  $B_4$ . We begin with

$$\frac{\varepsilon}{2} \int \partial_t \eta |V_j|^2 \leq \varepsilon \|\partial_t \eta\|_{L^\infty} \|V_j\|_{L^2}^2 \leq \frac{\varepsilon}{2} U_j^2 \|\partial_t \eta\|_{L^\infty}$$

We have that:

$$\begin{aligned} \|\partial_t \eta\|_{L^\infty} &= \left\| (I - b\varepsilon\Delta)^{-1} [(I + a\varepsilon\Delta) \operatorname{div} V + \varepsilon \operatorname{div}(\eta(W + V))] \right\|_{L^\infty} \\ &\leq \left\| (I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta) \operatorname{div} V \right\|_{L^\infty} + \varepsilon \left\| (I - b\varepsilon\Delta)^{-1} \operatorname{div}(\eta(W + V)) \right\|_{L^\infty} \\ &\leq \left\| (I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta) \operatorname{div} V \right\|_{B_{2,1}^{\frac{n}{2}}} + \varepsilon \|W\|_{B_{2,r}^s} + \varepsilon \|\eta V\|_{B_{2,r}^s} \end{aligned} \quad (2.30)$$

If  $b > 0$  or  $a = b = 0$ , then because the operator  $(I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta)$  maps  $L^2$  to  $L^2$  with norm independent of  $\varepsilon$ , we get that:

$$\|\partial_t \eta\|_{L^\infty} \leq C \max \left( 1, \frac{-a}{b} \right) (U_s + H U_s + U_s^2)$$

If  $b = 0$ ,  $a < 0$  then  $d > 0$  and we see that we have

$$-a\varepsilon \|\operatorname{div} \Delta V\|_{B_{2,1}^{\frac{d}{2}}} \leq -a\varepsilon \|\nabla^2 V\|_{B_{2,1}^{\frac{d}{2}+1}} \leq -a\varepsilon \|\nabla^2 V\|_{B_{2,1}^s} \leq \sqrt{\frac{-a}{d}} U_s. \quad (2.31)$$

We set

$$C_{abcd}^4 = \begin{cases} \max \left( 1, \frac{-a}{b} \right) & \text{if } b > 0 \text{ or } a = b = 0, \\ \max \left( 1, \sqrt{\frac{-a}{d}} \right) & \text{if } b = 0 \text{ and } a < 0. \end{cases} \quad (2.32)$$

Thus, we get that

$$\frac{\varepsilon}{2} \int \partial_t \eta |V_j|^2 \leq \varepsilon C C_{abcd}^4 U_j^2 (U_s + H U_s + U_s^2). \quad (2.33)$$

In a similar fashion we can treat the second term of  $B_4$  thus obtaining

$$\varepsilon^2 \frac{d}{2} \int \partial_t \eta \nabla V_j : \nabla V_j \leq C \frac{d}{d-a} C_{abcd}^4 U_j^2 (U_s + H U_s + U_s^2), \quad (2.34)$$

and we set

$$C_{abcd}^5 = \frac{d}{d-a} C_{abcd}^4. \quad (2.35)$$

Finally, the last term of  $B_4$  is estimated as follows:

$$-\varepsilon^2 d \langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \rangle_{L^2} \leq \varepsilon^2 d \|\nabla \eta\|_{L^\infty} \|\partial_t \nabla V_j\|_{L^2} \|V_j\|_{L^2},$$

and using

$$\partial_t \nabla V_j = -(I - \varepsilon d \Delta)^{-1} \nabla (\nabla \eta_j + c \varepsilon \nabla \Delta \eta_j + \varepsilon \nabla V_j W + \varepsilon \nabla V_j V - \varepsilon R_{2j})$$

we observe that

$$\begin{aligned} \varepsilon d \|\partial_t \nabla V_j\|_{L^2} &\leq \sqrt{d} (\|\eta_j\|_{L^2} - c \varepsilon \|\nabla \eta_j\|_{L^2} + \varepsilon \|W\|_{L^\infty} \|\nabla V_j\|_{L^2} \\ &\quad + \varepsilon \|V\|_{L^\infty} \|\nabla V_j\|_{L^2} + \varepsilon \|R_{2j}\|_{L^2}) \\ &\leq C C_{abcd}^6 (U_j (1 + H + U_s) + c_j^2(t) (H^2 + H U_s + U_s^2)), \end{aligned}$$

where

$$C_{abcd}^6 = \max \left( \sqrt{d}, \frac{-c \sqrt{d}}{\operatorname{sgn}(d) \sqrt{b-c}}, \sqrt{\frac{d}{d-a}} \right), \quad (2.36)$$

thus, we conclude that

$$\begin{aligned} -\varepsilon^2 d \langle \partial_t \nabla V_j, \nabla \eta \otimes V_j \rangle_{L^2} &\leq \varepsilon C C_{abcd}^6 U_j (U_j (U_s + H U_s + U_s^2) + \\ &\quad + c_j^2(t) U_s (H^2 + H U_s + U_s^2)). \end{aligned} \quad (2.37)$$

Combining estimates (2.33), (2.34), (2.37) we obtain:

$$B_4 \leq \varepsilon C \tilde{C}_{abcd} U_j \{U_j (U_s + H U_s + U_s^2) + c_j^9(t) U_s (H^2 + H U_s + U_s^2)\}, \quad (2.38)$$

where

$$\tilde{C}_{abcd} = \max_{i=1,6} C_{abcd}^i \quad (2.39)$$

is the maximum of all constants depending on  $a, b, c, d$  that appear in relations (2.17), (2.25), (2.28), (2.32), (2.35), (2.36).

Thus, supposing that we are not in the case:

$$a = d = 0 \text{ and } c < 0, \ b > 0,$$

we are able to successfully bound the  $T_i$ 's.

Finally, after adding up the estimations (2.18), (2.22), (2.24), (2.27), (2.29), (2.38) we obtain that there exists a positive constant  $C > 0$  depending only on  $n$  and  $s$  but not on  $\varepsilon$  such that

$$\frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon (b-c) |\nabla \eta_j|^2 + \varepsilon^2 (-c) b (\nabla^2 \eta_j : \nabla^2 \eta_j) \right)$$



$$\begin{aligned}
 & + \frac{1}{2} \partial_t \left( \int (1 + \varepsilon \eta) V_j^2 + \varepsilon (d - a + d\varepsilon \eta) (\nabla V_j : \nabla V_j) + \varepsilon^2 (-a) d (\nabla^2 V_j : \nabla^2 V_j) \right) \\
 & \leq \varepsilon C \tilde{C}_{abcd} U_j (U_j (U_s + H U_s + U_s^2) + c_j(t) (1 + U_s) (H^2 + H U_s + U_s^2)), \quad (2.40)
 \end{aligned}$$

where  $(c_j(t))_j$  is a sequence with  $(2^{js} c_j(t))_j \in \ell^r(\mathbb{Z})$ , having norm 1 and  $\tilde{C}_{abcd}$  is defined in (2.39). Actually for the sake of simplicity, from now on we will not carry on the distinction between constants that depend on the  $a, b, c, d$  parameters and the ones depending on the dimension  $n$  and regularity index  $s$ .

### 2.3 Another useful estimation

At this point, working with the non-cavitation hypothesis:

$$1 + \varepsilon \eta_0(x) \geq \alpha > 0, \quad (2.41)$$

using estimate (2.16) and a bootstrap argument would be sufficient in order to obtain long time existence result similar to the one obtained in [12] (with some restriction on the value of  $\varepsilon$  depending upon the initial data and  $\alpha$ ). However, proceeding in a slightly different manner we can avoid the use of (2.41) (although, all physically relevant data will verify it as  $1 + \varepsilon \eta$  represents the total height of the water over the flat bottom). We also stress out that the estimates established in this section are only available for the parameters  $a, b, c, d$  verifying (1.4) with the exception of the two cases (1.8). Let us investigate the following quantity:

$$\partial_t \left( \frac{\varepsilon}{2} \|\eta\|_{L^\infty} U_j^2 \right) = I_1 + I_2,$$

where

$$\begin{cases} 2I_1 = \varepsilon \|\eta\|_{L^\infty} \partial_t U_j^2, \\ 2I_2 = \varepsilon U_j^2 \partial_t \|\eta\|_{L^\infty}. \end{cases}$$

Owing to (2.11) we see that:

$$\begin{aligned}
 I_1 &= \frac{1}{2} \varepsilon \|\eta\|_{L^\infty} (T_1 + T_2 + T_3) + \frac{1}{2} \varepsilon \|\eta\|_{L^\infty} \left( -\varepsilon \int \eta \eta_j \operatorname{div} V_j + c\varepsilon^2 \int \eta \nabla \operatorname{div} V_j \nabla \eta_j \right) \\
 &\leq C\varepsilon^2 U_j U_s (U_j U_s + c_j(t) (H^2 + H U_s + U_s^2)) + C\varepsilon^2 \|\eta\|_{L^\infty}^2 \|\eta_j\|_{L^2} \|\operatorname{div} V_j\|_{L^2} \\
 &\quad + C\varepsilon^3 \|\eta\|_{L^\infty}^2 \|\nabla \operatorname{div} V_j\|_{L^2} \|\nabla \eta_j\|_{L^2} \\
 &\leq C\varepsilon^2 U_j U_s (U_j U_s + c_j(t) (H^2 + H U_s + U_s^2)) + C\varepsilon^{\frac{3}{2}} U_j^2 U_s^2 + C\varepsilon^{\frac{3}{2}} U_j^2 U_s^2 \\
 &\leq C\varepsilon U_j (U_j U_s^2 + c_j(t) U_s (H^2 + H U_s + U_s^2)). \quad (2.42)
 \end{aligned}$$

**Remark 2.1.** Let us notice that the term  $c\varepsilon^2 \int \eta \nabla \operatorname{div} V_j \nabla \eta_j$  raises some important issues. In order to successfully estimate it (and thus in order to have the validity of (2.42)), we need the restriction (1.8) on the parameters  $a, b, c, d$ . The idea is that when  $c \neq 0$ , we must have<sup>9</sup>

$$\operatorname{sgn}(b) + \operatorname{sgn}(d) - \operatorname{sgn}(a) \geq 2.$$

In view of the fact that  $b + d > 0$ , it transpires that we must exclude the cases:

$$a = d = 0, \quad b > 0, \quad c < 0 \quad \text{and}$$

---

<sup>9</sup>One of the unknown functions  $\eta, V$  must have at least  $B_{2,r}^{s+2}$ -regularity level while the other one needs  $B_{2,r}^{s+1}$ -regularity level.

$$a = b = 0, \quad d > 0, \quad c < 0.$$

Also, it is worth announcing that establishing (2.42) isn't the only place where the restriction on the parameters is needed. As it will be soon revealed, in order to obtain local existence of solutions we will again have to bound  $-c\varepsilon^2 \int \eta \nabla \operatorname{div} V_j \nabla \eta_j$  and the above considerations will have to apply.

In order to handle  $I_2$  we use the fact that the function  $t \rightarrow \|\eta(t)\|_{L^\infty}$  is locally Lipschitz we get that a.e.  $\partial_t \|\eta(t)\|_{L^\infty}$  exists and besides, a.e. in time we have

$$\begin{aligned} |\partial_t \|\eta(t)\|_{L^\infty}| &= \left| \lim_{s \rightarrow t} \frac{\|\eta(t)\|_{L^\infty} - \|\eta(s)\|_{L^\infty}}{t - s} \right| \leq \lim_{s \rightarrow t} \left\| \frac{\eta(t) - \eta(s)}{t - s} \right\|_{L^\infty} \\ &\leq \|\partial_t \eta(t)\|_{L^\infty} \leq \|\partial_t \eta(t)\|_{B_{2,1}^{\frac{n}{2}}}. \end{aligned}$$

Thus, owing to (2.30), (2.31) we get that:

$$I_2 \leq \frac{1}{2} \varepsilon U_j^2 \partial_t \|\eta\|_{L^\infty} \leq C \varepsilon U_j^2 (U_s + H U_s + U_s^2). \quad (2.43)$$

Combining (2.42) and (2.43) we get that

$$\partial_t \left( \frac{\varepsilon}{2} \|\eta\|_{L^\infty} U_j^2 \right) \leq C \varepsilon U_j (U_j (U_s + H U_s + U_s^2) + c_j(t) U_s (H^2 + H U_s + U_s^2)). \quad (2.44)$$

Finally let us observe that by adding (2.16) with (2.44) we obtain that:

$$\partial_t N_j^2 \leq C \varepsilon U_j (U_j (U_s + H U_s + U_s^2) + c_j(t) (1 + U_s) (H^2 + H U_s + U_s^2)) \quad (2.45)$$

where  $N_j$  is the quantity defined in (1.18).

## 2.4 The case $b > 0$ , $c < 0$ and $a = d = 0$

As pointed out in Section 2.2, we are not able to establish (2.16) for the case (2.15). However, if we proceed slightly different we can repair this inconvenience. Let us give some details of this aspect. As we have seen, the problem comes when estimating  $[\eta, \Delta_j] \nabla \operatorname{div} V$  (see (2.23)). In order to bypass this problem, let us rewrite equation (2.6) in the following manner

$$\begin{aligned} &\frac{-c\varepsilon}{2} \partial_t \left( \int (|\nabla \eta_j|^2 + \varepsilon b \nabla^2 \eta_j : \nabla^2 \eta_j) \right) - c\varepsilon \int \nabla \operatorname{div} V_j \nabla \eta_j \\ &\quad - c\varepsilon^2 \int \Delta_j (\eta \nabla \operatorname{div} V) \nabla \eta_j = T_5 \end{aligned} \quad (2.46)$$

where

$$\begin{aligned} T_5 &= c\varepsilon^2 \int (\nabla^2 \eta_j W : \nabla \eta_j) + c\varepsilon^2 \int (\nabla^2 \eta_j V : \nabla \eta_j) \\ &\quad + c\varepsilon^2 \left( \int \operatorname{div} V_j \nabla \eta \nabla \eta_j + \int \nabla V_j \nabla \eta \nabla \eta_j \right) + c\varepsilon^2 \int \Delta_j (\nabla W \nabla \eta) \nabla \eta_j - c\varepsilon^2 \int \tilde{R}_{j3} \nabla \eta_j \end{aligned}$$

and

$$\tilde{R}_{j3} = [W, \Delta_j] \nabla^2 \eta + [V, \Delta_j] \nabla^2 \eta + [\nabla \eta, \Delta_j] \operatorname{div} V + [\nabla \eta, \Delta_j] \nabla V.$$

We add (2.46) with (2.3) and (2.12) in order to obtain

$$\frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon(b - c) |\nabla \eta_j|^2 + \varepsilon^2 b (-c) \nabla^2 \eta_j : \nabla^2 \eta_j + (1 + \varepsilon \eta) |V_j|^2 \right)$$

$$+c\varepsilon^2 \int \eta \nabla \Delta \eta_j V_j - c\varepsilon^2 \int \Delta_j (\eta \nabla \operatorname{div} V) \nabla \eta_j = \varepsilon \int \nabla \eta \eta_j V_j + T_1 + T_4 + T_5.$$

Let us write that:

$$\begin{aligned} c\varepsilon^2 \int \eta \nabla \Delta \eta_j V_j &= -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j - c\varepsilon^2 \int \eta \Delta \eta_j \operatorname{div} V_j \\ &= -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j - c\varepsilon^2 \int \Delta \eta_j [\eta, \Delta_j] \operatorname{div} V - c\varepsilon^2 \int \Delta \eta_j \Delta_j (\eta \operatorname{div} V) \\ &= -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j - c\varepsilon^2 \int \Delta \eta_j [\eta, \Delta_j] \operatorname{div} V + c\varepsilon^2 \int \nabla \eta_j \Delta_j (\nabla \eta \operatorname{div} V) \\ &\quad + c\varepsilon^2 \int \nabla \eta_j \Delta_j (\eta \nabla \operatorname{div} V) \\ &= -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j - c\varepsilon^2 \int \Delta \eta_j [\eta, \Delta_j] \operatorname{div} V + c\varepsilon^2 \int \nabla \eta_j \nabla \eta \operatorname{div} V_j \\ &\quad + c\varepsilon^2 \int \nabla \eta_j [\Delta_j, \nabla \eta] \operatorname{div} V + c\varepsilon^2 \int \nabla \eta \Delta_j (\eta \nabla \operatorname{div} V). \end{aligned}$$

Thus, we get:

$$\begin{aligned} &c\varepsilon^2 \int \eta \nabla \Delta \eta_j V_j - c\varepsilon^2 \int \nabla \eta \Delta_j (\eta \nabla \operatorname{div} V) \\ &= -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j - c\varepsilon^2 \int \Delta \eta_j [\eta, \Delta_j] \operatorname{div} V + c\varepsilon^2 \int \nabla \eta_j \nabla \eta \operatorname{div} V_j \\ &\quad + c\varepsilon^2 \int \nabla \eta_j [\Delta_j, \nabla \eta] \operatorname{div} V \\ &= -c\varepsilon^2 \int \nabla \eta \Delta \eta_j V_j - c\varepsilon^2 \int \Delta \eta_j [\eta, \Delta_j] \operatorname{div} V - c\varepsilon^2 \int \nabla^2 \eta_j \nabla \eta V_j \\ &\quad - c\varepsilon^2 \int \nabla^2 \eta \nabla \eta_j V_j + \int \nabla \eta_j [\Delta_j, \nabla \eta] \operatorname{div} V \\ &\leq -c\varepsilon^2 \left( \|\nabla \eta\|_{L^\infty} \|\Delta \eta_j\|_{L^2} \|V_j\|_{L^2} + \|\Delta \eta_j\|_{L^2} c_j(t) \|\nabla \eta\|_{B_{2,r}^{s-1}} \|V\|_{B_{2,r}^s} + \right. \\ &\quad \left. + \|\nabla \eta\|_{L^\infty} \|\nabla^2 \eta_j\|_{L^2} \|V_j\|_{L^2} + \|\nabla^2 \eta\|_{L^\infty} \|\nabla \eta_j\|_{L^2} \|V_j\|_{L^2} + \right. \\ &\quad \left. + \|\nabla \eta_j\|_{L^2} c_j(t) \|\nabla^2 \eta\|_{B_{2,r}^{s-1}} \|V\|_{B_{2,r}^s} \right) \\ &\leq -cC\varepsilon (U_j^2 U_s + c_j(t) U_j U_s^2). \end{aligned} \tag{2.47}$$

A similar reasoning as in (2.23) together with estimates (2.22), (2.29), (2.38) and (2.47) gives us :

$$\begin{aligned} &\frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon (b-c) |\nabla \eta_j|^2 + \varepsilon^2 (-c)b (\nabla^2 \eta_j : \nabla^2 \eta_j) + (1+\varepsilon \eta) V_j^2 \right) \\ &\leq C\varepsilon U_j (U_j (U_s + H U_s + U_s^2) + c_j(t) (1+U_s) (H^2 + U_s H + U_s^2)). \end{aligned} \tag{2.48}$$

### 3 Existence and uniqueness

We begin by defining what kind of solutions we are looking for:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + a\varepsilon \operatorname{div} \Delta V + \varepsilon W \nabla \eta + \varepsilon \operatorname{div} (\eta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + c\varepsilon \nabla \Delta \eta + \varepsilon \frac{1}{2} \nabla |W|^2 + \varepsilon \nabla W V + \varepsilon \nabla V W + \varepsilon \frac{1}{2} \nabla |V|^2 = 0, \\ \eta|_{t=0} = \eta_0, \quad V|_{t=0} = V_0. \end{cases}$$

**Definition 3.1.** Let us consider a positive time  $T > 0$  and  $(\eta_0, V_0) \in L^2 \times (L^2)^n$  and  $W \in (H^1(\mathbb{R}^n))^n$ . A pair  $(\eta, V) \in \mathcal{C}([0, T], L^2 \times (L^2)^n)$  is called a solution to (1.10) on  $[0, T]$  if for any  $(\phi, \psi) \in \mathcal{C}^1([0, T], \mathcal{S} \times \mathcal{S}^n)$  and for all  $t \in [0, T]$ , the following identities hold true:

$$\begin{aligned} & \int_0^t \langle \eta, (I - \varepsilon b \Delta) \partial_t \phi \rangle_{L^2} + \int_0^t \langle V, (I + a \varepsilon \Delta) \nabla \phi \rangle_{L^2} + \varepsilon \int_0^t \langle \eta, \nabla (W \phi) \rangle_{L^2} + \varepsilon \int_0^t \langle \eta V, \nabla \phi \rangle_{L^2} \\ &= \langle \eta(t), (I - \varepsilon b \Delta) \phi(t) \rangle_{L^2} - \langle \eta_0, (I - \varepsilon b \Delta) \phi(0) \rangle_{L^2} \end{aligned}$$

and

$$\begin{aligned} & \int_0^t \langle V, (I - \varepsilon d \Delta) \partial_t \psi \rangle_{L^2} + \int_0^t \langle \eta, (I + c \varepsilon \Delta) \operatorname{div} \psi \rangle_{L^2} + \varepsilon \int_0^t \left\langle \frac{|W|^2}{2}, \operatorname{div} \psi \right\rangle_{L^2} + \\ & \varepsilon \int_0^t \langle V, \nabla^t W \psi \rangle_{L^2} + \varepsilon \int_0^t \langle V, \operatorname{div} (\psi \otimes W) \rangle_{L^2} + \varepsilon \int_0^t \left\langle \frac{|V|^2}{2}, \operatorname{div} \psi \right\rangle_{L^2} \\ &= \langle V(t), (I - \varepsilon b \Delta) \psi(t) \rangle_{L^2} - \langle V_0, (I - \varepsilon b \Delta) \psi(0) \rangle_{L^2}. \end{aligned}$$

Let us state now the following local existence and uniqueness theorem which serves as an intermediary result for Theorem 1:

**Theorem 2.** Let  $a, b, c, d$  be chosen as in (1.4) excluding the two cases (1.8),  $b + d > 0$ ,  $r \in [0, \infty]$  and  $s \in \mathbb{R}$  such that:

$$s > \frac{n}{2} + 1 \text{ or } s = \frac{n}{2} + 1 \text{ and } r = 1. \quad (3.1)$$

Furthermore, let us consider  $s_1, s_2$  and  $s_3$  defined by relation (1.12) and  $W \in (B_{2,r}^{s_3})^n$ . Then, for all  $(\eta_0, V_0) \in B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n$  with  $\operatorname{curl} V_0 = 0$ , there exists a positive  $T > 0$  and an unique solution

$$\begin{aligned} & (\eta, V) \in \mathcal{C}([0, T], B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n) \text{ if } r < \infty \text{ or} \\ & (\eta, V) \in L^\infty([0, T], B_{2,\infty}^{s_1} \times (B_{2,\infty}^{s_2})^n) \cap \bigcap_{\beta > 0} \mathcal{C}([0, T], B_{2,\infty}^{s_1-\beta} \times (B_{2,\infty}^{s_2-\beta})^n) \text{ if } r = \infty, \end{aligned}$$

of equation (1.10). Moreover, if we denote by  $T(\eta_0, V_0)$  the maximal time of existence then, if  $T(\eta_0, V_0) < \infty$ , we have that:

$$\lim_{t \rightarrow T(\eta_0, V_0)} U_s(t) = \infty \text{ if } r < \infty, \quad (3.2)$$

$$\limsup_{t \rightarrow T(\eta_0, V_0)} U_s(t) = \infty \text{ if } r = \infty. \quad (3.3)$$

Of course, the local existence result of the above theorem is not optimal. For some particular choices of the  $a, b, c, d$  parameters we can solve the Cauchy problem for initial data in larger spaces, see for instance [2], [5], [6]. However, the lower bound on the time of existence is at most of order  $O(\varepsilon^{-\frac{1}{2}})$ . Initially, this is also the case of the solution constructed in Theorem 2. As mentioned above this time scale is not satisfactory from a practical point of view. However, as a by-product of the explosion criteria (3.2)-(3.3) of the solution of (1.10) and some refined energy estimate, we can improve the lower bound of the  $T(\eta_0, V_0)$  thus establishing  $O(\frac{1}{\varepsilon})$ -long time existence.

Having established all the estimates that we need, let us proceed by proving Theorem 2.

*Proof.* We will use the so called Friedrichs method. For all  $m \in \mathbb{N}$ , let us consider  $\mathbb{E}_m$  the low frequency cut-off operator defined by:

$$\mathbb{E}_m f = \mathcal{F}^{-1} \left( \chi_{B(0,m)} \hat{f} \right).$$

We define the space

$$L_m^2 = \left\{ f \in L^2 : \text{Supp } \hat{f} \subset B(0,m) \right\}$$

which, endowed with the  $\|\cdot\|_{L^2}$ -norm is a Banach space. Let us observe that due to Bernstein's lemma, all Sobolev norms are equivalent on  $L_m^2$ . For all  $m \in \mathbb{N}$ , we consider the following differential equation on  $L_m^2$ :

$$\begin{cases} \partial_t \eta = F_m(\eta, V), \\ \partial_t V = G_m(\eta, V), \\ \eta|_{t=0} = \mathbb{E}_m \eta_0, \quad V|_{t=0} = \mathbb{E}_m V_0, \end{cases} \quad (3.4)$$

where  $(F_m, G_m) : L_m^2 \times (L_m^2)^n \rightarrow L_m^2 \times (L_m^2)^n$  are defined by:

$$F_m(\eta, V) = -\mathbb{E}_m \left( (I - \varepsilon b \Delta)^{-1} [(I + a \varepsilon \Delta) \text{div } V + \varepsilon W \nabla \eta + \varepsilon \text{div}(\eta V)] \right), \quad (3.5)$$

$$G_m(\eta, V) = -\mathbb{E}_m \left( (I - \varepsilon d \Delta)^{-1} \left[ (I + c \varepsilon \Delta) \nabla \eta + \frac{\varepsilon}{2} \nabla |V + W|^2 \right] \right). \quad (3.6)$$

It transpires that due to the equivalence of the Sobolev norm,  $(F_m, G_m)$  is continuous and locally Lipschitz on  $L_m^2 \times (L_m^2)^n$ . Thus, the classical Picard theorem ensures that there exists a nonnegative time  $T_m > 0$  and a unique solution  $(\eta^m, V^m) : \mathcal{C}^1([0, T_m], L_m^2 \times (L_m^2)^n)$ . Moreover, if  $T_m$  is finite, then:

$$\lim_{t \rightarrow T_m} \|(\eta^m, V^m)\|_{L^2} = \infty.$$

Another important aspect is that because of the property  $\mathbb{E}_m^2 = \mathbb{E}_m$  we get that the estimate obtained in (2.11) still holds true for  $(\eta_m, V_m)$ , namely

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \int (\eta_j^m)^2 + \varepsilon (b - c) |\nabla \eta_j^m|^2 + \varepsilon^2 (-c) b (\nabla^2 \eta_j^m : \nabla^2 \eta_j^m) \right) \\ & + \frac{1}{2} \partial_t \left( \int |V_j^m|^2 + \varepsilon (d - a) (\nabla V_j^m : \nabla V_j^m) + \varepsilon^2 (-a) d (\nabla^2 V_j^m : \nabla^2 V_j^m) \right) \\ & + \varepsilon \int \eta^m \eta_j^m \text{div } V_j^m - c \varepsilon^2 \int \eta^m \nabla \text{div } V_j^m \nabla \eta_j^m = T_1 + T_2 + T_3 \end{aligned} \quad (3.7)$$

with  $T_1, T_2, T_3$  defined as in relations (2.4), (2.7), (2.10) but with  $(\eta^m, V^m)$  instead of  $(\eta, V)$ . Considering  $U_j^m$  and  $U_s^m$  the quantities defined in (1.16) and (1.17) with  $(\eta^m, V^m)$  instead of  $(\eta, V)$ . Also, for the  $T_i$ 's we dispose of the estimates (2.22), (2.24) and (2.27) with  $U_j^m$  and  $U_s^m$  instead of  $U_j$  and  $U_s$  and thus we gather that:

$$T_1 + T_2 + T_3 \leq \varepsilon C U_j^m \left( U_j^m U_s^m + c_j(t) \left( H^2 + H U_s^m + (U_s^m)^2 \right) \right).$$

Also, let us notice that<sup>10</sup>:

$$-\varepsilon \int \eta^m \eta_j^m \text{div } V_j^m \leq \varepsilon^{\frac{1}{2}} C (U_j^m)^2 U_s^m.$$

<sup>10</sup>Observe that due to (1.12) at least one of  $\eta, V$  has regularity level  $B_{2,r}^{s+1}$  and thus, eventually by an integration by parts we can obtain the announced estimate.

Next, observe that<sup>11</sup>:

$$c\varepsilon^2 \int \eta^m \nabla \operatorname{div} V_j^m \nabla \eta_j^m \leq \varepsilon^{\frac{1}{2}} C (U_j^m)^2 U_s^m.$$

We thus gather that:

$$\frac{1}{2} \partial_t (U_j^m)^2 \leq \varepsilon^{\frac{1}{2}} C U_j^m \left( U_j^m U_s^m + c_j(t) \left( H^2 + H U_s^m + (U_s^m)^2 \right) \right)$$

and by a Gronwall-type argument we obtain that for all  $t \in [0, T_m]$ :

$$U_j^m(t) \leq U_j^m(0) + \varepsilon^{\frac{1}{2}} C \int_0^t \left( U_j^m U_s^m + c_j(\tau) \left( H^2 + H U_s^m + (U_s^m)^2 \right) \right) d\tau,$$

thus multiplying with  $2^{js}$  and performing an  $\ell^r(\mathbb{Z})$ -summation, owing to Minkowsky's theorem, we get that:

$$U_s^m(t) \leq U_s^m(0) + C\varepsilon^{\frac{1}{2}} t + \varepsilon^{\frac{1}{2}} C \max(1, H) \int_0^t \left( U_s^m + (U_s^m)^2 \right) d\tau.$$

We denote by  $H^* = \max(1, H)$ . Thus, Gronwall's lemma along with the explosion criterion for ODE's gives us that:

$$\frac{\ln \left( 1 + \frac{1}{U_s^m(0)} \right)}{\varepsilon^{\frac{1}{2}} C H^*} \leq T_m^*$$

where  $T_m^*$  is the maximal time of existence for (3.4). Also, due to the nature of the cut-off operator  $\mathbb{E}_m$ , it transpires that:

$$U_s^m(0) \leq U_s(0),$$

and consequently that:

$$\frac{\ln \left( 1 + \frac{1}{U_s^m(0)} \right)}{\varepsilon^{\frac{1}{2}} C H^*} \leq T_m^*. \quad (3.8)$$

In particular, for each time  $T > 0$  such that the above inequality is strictly satisfied we obtain due to Gronwall's lemma that for all  $t \in [0, T]$

$$U_s^m(t) \leq \frac{1}{e^{\ln(1 + \frac{1}{U_s^m(0)}) - \varepsilon^{\frac{1}{2}} C H^* T} - 1} \quad (3.9)$$

and thus, the solution  $(\eta_m, V_m)$  is uniformly bounded on  $[0, T]$ . Next, from relations (3.4) and (3.5) we get that  $(\partial_t \eta^m)_{m \in \mathbb{N}}$  is uniformly bounded on  $[0, T]$  in  $B_{2,r}^{s-1}$ . Indeed, the first term is

$$(I - b\varepsilon\Delta)^{-1} (I + a\varepsilon\Delta) \operatorname{div} V^m \in B_{2,r}^{s_4}$$

with<sup>12</sup>

$$s_4 = s + 2\operatorname{sgn}(b) + \operatorname{sgn}(d) + \operatorname{sgn}(a) - 1 \geq s - 1.$$

<sup>11</sup>Again, because of (1.12) if  $c \neq 0$  then at least one of  $\eta, V$  has regularity level  $B_{2,r}^{s+2}$  while the other one has regularity level  $B_{2,r}^{s+1}$  see Remark (2.1). Thus, eventually by an integration by parts, we get the announced result.

<sup>12</sup>remember that  $b + d > 0$

In view of (3.1),  $B_{2,r}^{s-1}$  is an algebra and thus the last two terms of (3.5) are also at least in  $B_{2,r}^{s-1}$ . Thus, in view of the uniform estimates (3.9), we get that  $(\partial_t \eta^m)_{m \in \mathbb{N}}$  is uniformly bounded on  $[0, T]$  in  $B_{2,r}^{s-1}$ . Next, considering for all  $p \in \mathbb{N}$  a smooth function  $\phi_p$  such that:

$$\begin{cases} \text{Supp}(\phi_p) \subset B(0, p+1) \\ \phi_p = 1 \text{ on } B(0, p) \end{cases}$$

it follows, in view of Proposition 6.5 that for each  $p \in \mathbb{N}$ , the sequence  $(\phi_p \eta^m)_{m \in \mathbb{N}}$  is uniformly equicontinuous on  $[0, T]$  and that for all  $t \in [0, T]$ , the set  $\{\phi_p \eta^m(t) : m \in \mathbb{N}\}$  is relatively compact in  $B_{2,r}^{s-1}$ . Thus, the Ascoli-Arzelà Theorem combined with Cantor's diagonal process provides us a subsequence of  $(\eta^m)_{m \in \mathbb{N}}$ <sup>13</sup> and a tempered distribution  $\eta \in \mathcal{C}([0, T], \mathcal{S}')$  such that for all  $\phi \in \mathcal{D}(\mathbb{R}^n)$ :

$$\phi \eta^m \rightarrow \phi \eta \text{ in } \mathcal{C}([0, T], B_{2,r}^{s-1}).$$

Moreover, owing to Proposition 6.3 and (3.9) we get that  $\eta \in L_T^\infty(B_{2,r}^{s_1})$  and using interpolation we get that

$$\phi \eta^m \rightarrow \phi \eta \text{ in } \mathcal{C}([0, T], B_{2,r}^{s_1-\gamma})$$

for all  $\gamma > 0$ . Of course, by the same argument one can get  $V \in L_T^\infty((B_{2,r}^{s_2})^n)$  such that for all  $\psi \in (\mathcal{D}(\mathbb{R}^n))^n$ :

$$\psi V^m \rightarrow \psi V \text{ in } \mathcal{C}([0, T], (B_{2,r}^{s_2-\gamma})^n)$$

for all  $\gamma > 0$ . We claim that the properties enlisted above permit us to pass to the limit when  $m \rightarrow \infty$  in the equation verified  $\eta^m$  and  $V^m$ . By the Fatou property of Besov spaces we get that  $(\eta, V) \in L_t^\infty(B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n)$ . It remains to verify that  $(\eta, V)$  has the announced regularity. Suppose that  $r < \infty$ . From  $\eta$ 's equation we see that  $\partial_t \eta \in L_t^\infty(B_{2,r}^{s-1})$  and thus,  $\eta \in \mathcal{C}([0, T], B_{2,r}^{s-1})$  which also implies that  $S_j \eta \in \mathcal{C}([0, T], B_{2,r}^{s_1})$  for all  $j \in \mathbb{Z}$ . The conclusion follows as the sequence of  $B_{2,r}^{s_1}$ -valued functions  $(S_j \eta)_{j \in \mathbb{Z}}$  tends uniformly to  $\eta$ . Indeed, for all  $\ell, j \in \mathbb{Z}$ , we have

$$\Delta_\ell(\eta - S_j \eta) = 0 \text{ if } \ell \leq j - 2,$$

therefore we have

$$\begin{aligned} \|\eta - S_j \eta\|_{L_t^\infty(B_{2,r}^{s_1})} &\leq \left( \sum_{\ell \geq j-1} 2^{\ell r s_1} \|\eta_0\|_{L^2}^r \right)^{\frac{1}{r}} \\ &\quad + \varepsilon^{\frac{1}{2}} C \left( \|U_s\|_{L_t^\infty(B_{2,r}^{s_1})}^2 + H^2 \right) \int_0^t \left( \sum_{\ell \geq j-1} 2^{\ell r s_1} (U_j(\tau) + c_j(\tau))^r \right)^{\frac{1}{r}} \end{aligned}$$

and as a consequence of Proposition 6.3 and the dominated convergence theorem we find that

$$\lim_{j \rightarrow \infty} \|\eta - S_j \eta\|_{L_t^\infty(B_{2,r}^{s_1})} = 0,$$

which implies that  $\eta \in \mathcal{C}([0, T], B_{2,r}^{s_1})$ . A similar argument shows that  $V \in \mathcal{C}([0, T], (B_{2,r}^{s_2})^n)$ . When  $r = \infty$ , we know that for every positive  $\beta$ ,  $B_{2,1}^{s_1-\beta}$  is continuously embedded in  $B_{2,\infty}^{s_1}$  and repeating the above argument permits us to conclude that  $(\eta, V)$  has the desired regularity. This completes the

<sup>13</sup>Still denoted  $(\eta^m)_{m \in \mathbb{N}}$  for the sake of simplicity.

proof of existence. Uniqueness, is a consequence of the following stability estimate: let us consider two solutions of (1.10),  $(\eta^1, V^1)$ ,  $(\eta^2, V^2)$  and observe that the difference

$$(\delta\eta, \delta V) = (\eta^1 - \eta^2, V^1 - V^2)$$

satisfies the following system:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \delta\eta + \operatorname{div} \delta V + a\varepsilon \operatorname{div} \Delta \delta V + \varepsilon W \nabla \delta\eta + \varepsilon \operatorname{div} (\delta\eta V^1) + \varepsilon \operatorname{div} (\eta^2 \delta V) = 0 \\ (I - \varepsilon d \Delta) \partial_t \delta V + \nabla \delta\eta + c\varepsilon \nabla \Delta \delta\eta + \varepsilon \nabla W (\delta V) + \varepsilon \nabla (\delta V) W + \varepsilon \nabla (\delta V) V^1 + \varepsilon \nabla V^2 \delta V = 0 \\ \delta\eta|_{t=0} = 0, \delta V|_{t=0} = 0. \end{cases} \quad (3.10)$$

We consider  $U_s^1 = U_s(\eta^1, V^1)$  and  $U_s^2 = U_s(\eta^2, V^2)$ , see (1.17). For the sake of simplicity we will prove stability estimates in the classical Sobolev space  $X = H^{r_1} \times (H^{r_2})^n$  with:

$$\begin{cases} r_1 = \operatorname{sgn}(b) - \operatorname{sgn}(c), \\ r_2 = \operatorname{sgn}(d) - \operatorname{sgn}(a). \end{cases}$$

We endow  $X$  with the norm:

$$\begin{aligned} \|\eta, V\|_X^2 &= \|\eta\|_{L^2}^2 + \varepsilon(b-c) \|\nabla \eta\|_{L^2}^2 - \varepsilon^2 bc \|\nabla^2 \eta\|_{L^2}^2 \\ &\quad + \|V\|_{L^2}^2 + \varepsilon(d-a) \|\nabla V\|_{L^2}^2 - \varepsilon^2 da \|\nabla^2 V\|_{L^2}^2 \end{aligned}$$

Observe that due to the fact that  $s_1, s_2$  are chosen so as to satisfy (1.12) with  $s$  chosen as in (3.1), we have that  $H^{r_1} \times (H^{r_2})^n$  is continuously embedded in  $B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n$ . Multiplying the first equation in (3.10) with  $\eta$ , the second equation with  $V$  and adding up the results we get that:

$$\begin{aligned} \frac{1}{2} \partial_t \left( \|\delta\eta\|_{L^2}^2 + \varepsilon(b-c) \|\nabla \delta\eta\|_{L^2}^2 + \|\delta V\|_{L^2}^2 + \varepsilon(d-a) \|\nabla \delta V\|_{L^2}^2 \right) &+ \varepsilon a \int \Delta \operatorname{div} \delta V \delta\eta \\ &+ \varepsilon c \int \Delta \nabla \delta\eta \delta V \leq C\sqrt{\varepsilon} (H + U_s^1 + U_s^2) \|\delta\eta, \delta V\|_X^2. \end{aligned}$$

Now, let us multiply the first equation in (3.10) with  $c\varepsilon \Delta \eta$ , the second equation with  $a\varepsilon \Delta V$  and adding up the results we obtain:

$$\begin{aligned} \frac{1}{2} \partial_t \left( -c\varepsilon \|\nabla \delta\eta\|_{L^2}^2 - \varepsilon^2 bc \|\nabla^2 \delta\eta\|_{L^2}^2 - \varepsilon a \|\nabla \delta V\|_{L^2}^2 - \varepsilon^2 da \|\nabla^2 \delta V\|_{L^2}^2 \right) \\ + \varepsilon a \int \nabla \delta\eta \Delta \delta V + \varepsilon c \int \operatorname{div} \delta V \Delta \delta\eta \\ \leq C\sqrt{\varepsilon} (H + U_s^1 + U_s^2) \|\delta\eta, \delta V\|_X^2. \end{aligned}$$

Thus, adding up the above relations gives us:

$$\partial_t \|\delta\eta, \delta V\|_X^2 \leq C\sqrt{\varepsilon} (H + U_s^1 + U_s^2) \|\delta\eta, \delta V\|_X^2.$$

Hence, Gronwall's lemma ensures the desired result.

Proving the blow-up criteria is classic. Let us suppose that  $T(\eta_0, V_0) < \infty$  and that

$$\limsup_{t \rightarrow T(\eta_0, V_0)} U_s(t) < \infty.$$

Then,  $U_s(t)$  remains bounded on  $[0, T(\eta_0, V_0))$  say:

$$U_s(t) \leq M,$$



for all  $t \in [0, T(\eta_0, V_0))$ . We see that for any  $t_0 \in [0, T(\eta_0, V_0))$  we can construct, using the same method as before a solution to (1.10) with initial data  $(\eta(t_0), V(t_0))$  on a time interval that according to (3.8) satisfies the following lower bound:

$$T^{new} - t_0 \geq \frac{\ln\left(1 + \frac{1}{U_s(t_0)}\right)}{\varepsilon^{\frac{1}{2}} CH^*} \geq \frac{\ln\left(1 + \frac{1}{M}\right)}{\varepsilon^{\frac{1}{2}} CH^*}.$$

Of course, choosing  $t_0$  close enough to  $T(\eta_0, V_0)$  we can obtain  $T^{new} > T(\eta_0, V_0)$  such that gluing together the new solution with the  $(\eta, V)_{|[0, t_0]}$  and in view of the uniqueness we get a contradiction on the maximality of  $T(\eta_0, V_0)$ . This concludes the proof of the announced result.  $\square$

## 4 The proof of Theorem 1

We are now in the position of establishing the announced long time existence result. Actually, we prove long time existence and uniqueness in the framework of the more general Besov space. More precisely, Theorem 1 is just a particular case of the following:

**Theorem 3.** *Let  $a, b, c, d$  as in (1.4) excluding the two cases (1.8),  $b + d > 0$ . Let us take  $r \in [1, \infty]$  and  $s$  such that*

$$s > \frac{n}{2} + 1 \text{ or } s = \frac{n}{2} + 1 \text{ and } r = 1,$$

*with  $n \geq 1$ . Let us also consider  $s_1, s_2$  and  $s_3$  defined by (1.12) and  $W \in (B_{2,r}^{s_3})^n$ . Then, we can establish long time existence and uniqueness of solutions (see Definition 1.2) for the equation (1.10) in  $B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n$ . Moreover, if we denote by  $T(\eta_0, V_0)$ , the maximal time of existence then there exists some  $T \in [0, T(\eta_0, V_0))$  which is bounded from below by an  $O(\frac{1}{\varepsilon})$ -order quantity and a function  $G : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $t \in [0, T]$  we have:*

$$U_s(\eta, V) \leq G(U_s(\eta_0, V_0))$$

where  $U_s(\eta, V)$  is defined in relation (1.17).

Let us consider the unique maximal solution  $(\eta, V)$  of (1.10) that we constructed in the proof of Theorem 2. Of course, the estimate (2.45) holds true for this solution thus we have that:

$$\partial_t N_j^2 \leq C\varepsilon U_j (U_j (U_s + HU_s + U_s^2) + c_j(t)(1 + U_s) (H^2 + U_s H + U_s^2)). \quad (4.1)$$

We recall that  $N_j$  is the following quantity:

$$\begin{aligned} N_j^2(t) &= \int (1 + \varepsilon \|\eta\|_{L^\infty}) \eta_j^2 + \varepsilon(b - c)(1 + \varepsilon \|\eta\|_{L^\infty}) |\nabla \eta_j|^2 \\ &\quad + \int \varepsilon^2(-c)b(1 + \varepsilon \|\eta\|_{L^\infty}) (\nabla^2 \eta_j : \nabla^2 \eta_j) \\ &\quad + \int (1 + \varepsilon \eta + \varepsilon \|\eta\|_{L^\infty}) V_j^2 + \varepsilon(d - a + d\varepsilon \eta + d\varepsilon \|\eta\|_{L^\infty}) (\nabla V_j : \nabla V_j) \\ &\quad + \int \varepsilon^2(-a)d(1 + \varepsilon \|\eta\|_{L^\infty}) (\nabla^2 V_j : \nabla^2 V_j). \end{aligned}$$

and that we have:

$$U_j(t) \leq N_j(t) \leq (1 + 2\varepsilon \|\eta(t)\|_{L^\infty})^{\frac{1}{2}} U_j(t) \quad (4.2)$$

thus we immediately get that

$$\partial_t N_j^2 \leq C\varepsilon N_j (U_j (U_s + HU_s + U_s^2) + c_j(t)(1 + U_s) (H^2 + U_s H + U_s^2))$$

and by time integration we get that:

$$U_j(t) \leq N_j(t) \leq N_j(0) + C\varepsilon \int_0^t (U_j (U_s + HU_s + U_s^2) + c_j(t)(1 + U_s) (H^2 + U_s H + U_s^2)) d\tau.$$

Multiplying the last inequality with  $2^{js}$  and performing a  $\ell^r(\mathbb{Z})$  summation yields:

$$\begin{aligned} U_s(t) &\leq N_0 + C\varepsilon \int_0^t (H^2 + U_s (H + H^2) + U_s^2 (1 + H) + U_s^3) d\tau, \\ &\leq N_0 + C\varepsilon t H^2 + \varepsilon C (1 + H + H^2) \int_0^t (U_s + U_s^3) d\tau, \\ &\leq N_0 + C\varepsilon t H^2 + \varepsilon C (1 + H^2) \int_0^t (U_s + U_s^3) d\tau, \end{aligned} \tag{4.3}$$

for all  $t \in [0, T(\eta_0, V_0))$  where

$$N_0 = \left\| (2^{js} N_j(0))_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} \leq (1 + 2\varepsilon \|\eta_0\|_{L^\infty})^{\frac{1}{2}} \|(\eta_0, V_0)\|_{B_{2,r}^s}.$$

Having established (4.3) we are in the position of finding a lower bound of order  $O(\frac{1}{\varepsilon})$  for the time of existence. Indeed by Gronwall's lemma, and taking in account the explosion criterion of Theorem 2 we get that, supposing  $T(\eta_0, V_0)$  is finite:

$$\limsup_{t \rightarrow T(\eta_0, V_0)} U_s(t) = +\infty. \tag{4.4}$$

Hence, via Gronwall's lemma, we have:

$$\frac{1}{2} \ln \left( 1 + \frac{1}{(N_0 + \varepsilon C H^2 T(\eta_0, V_0))^2} \right) \leq \varepsilon C (1 + H^2) T(\eta_0, V_0). \tag{4.5}$$

If

$$\varepsilon C H^2 T(\eta_0, V_0) \geq N_0 \text{ i.e. } T(\eta_0, V_0) \geq \frac{N_0}{\varepsilon C H^2},$$

then, we have nothing else to prove. If this is not the case, the LHS member of (4.5) is larger then  $\frac{1}{2} \ln \left( 1 + \frac{1}{4N_0^2} \right)$  and we get that:

$$T(\eta_0, V_0) \geq \frac{\frac{1}{2} \ln \left( 1 + \frac{1}{4N_0^2} \right)}{\varepsilon C (1 + H^2)}.$$

Thus,  $T(\eta_0, V_0)$  is always bounded from below by a quantity of order  $O(\frac{1}{\varepsilon})$ .

We now prove that on a  $O(\frac{1}{\varepsilon})$ -order time interval we dispose of uniform bounds for the solution of (1.10). Let us consider

$$T^* = \sup \{ T \in [0, T(\eta_0, V_0)) : \forall t \in [0, T], U_s(t) \leq 2(N_0 + \varepsilon C t H^2) \}.$$

Then, from (4.3), we deduce that for all  $t \leq T^*$  we have

$$U_s(t) \leq N_0 + \varepsilon C T^* H^2 + \varepsilon C (1 + H^2) \left( 1 + 4 (N_0 + \varepsilon C T^* H^2)^2 \right) \int_0^t U_s(\tau) d\tau$$

and according to Gronwall's lemma we get that:

$$U_s(t) \leq (N_0 + \varepsilon C T^* H^2) \exp \left( \varepsilon T^* C (1 + H^2) \left( 1 + 4 (N_0 + \varepsilon C T^* H^2)^2 \right) \right).$$

Now, if there exists a  $\beta \in (0, 2)$  such that

$$\exp \left( \varepsilon T^* C (1 + H^2) \left( 1 + 4 (N_0 + \varepsilon C T^* H^2)^2 \right) \right) \leq \beta,$$

then a continuity argument will lead us to the conclusion that  $T^* = T(\eta_0, V_0)$  which will imply that  $T(\eta_0, V_0) = \infty$ . Thus, for all  $t \in [0, \frac{N_0}{\varepsilon C H^2}]$  we get that

$$U_s(t) \leq 4N_0 \leq (1 + 2\varepsilon U_s(\eta_0, V_0))^{\frac{1}{2}} U_s(\eta_0, V_0). \quad (4.6)$$

Assume that

$$\varepsilon T^* C (1 + H^2) \left( 1 + 4 (N_0 + \varepsilon C T^* H^2)^2 \right) \geq \ln 2. \quad (4.7)$$

Then, if

$$T^* > \frac{N_0}{\varepsilon C H^2}, \quad (4.8)$$

we get that (4.6) holds true for all  $t \in [0, \frac{N_0}{\varepsilon C H^2}]$ . If (4.8) doesn't hold i.e.

$$T^* \leq \frac{N_0}{\varepsilon C H^2} \quad (4.9)$$

then, combining (4.7) with (4.9), we get that:

$$T^* \geq \frac{\ln 2}{\varepsilon C (1 + H^2) (1 + 16N_0^2)}$$

and thus, using the definition of  $T^*$ , we get that for all  $t \in [0, \frac{\ln 2}{\varepsilon C (1 + H^2) (1 + 16N_0^2)}]$

$$\begin{aligned} U_s(t) &\leq 2 \left( N_0 + \frac{H^2 \ln 2}{(1 + H^2) (1 + 16N_0^2)} \right), \\ &\leq 2 \left( N_0 + \frac{\ln 2}{(1 + 16N_0^2)} \right) \end{aligned} \quad (4.10)$$

$$\leq 2 \left( (1 + 2\varepsilon U_s(\eta_0, V_0))^{\frac{1}{2}} U_s(\eta_0, V_0) + \frac{\ln 2}{(1 + 16U_s^2(\eta_0, V_0))} \right). \quad (4.11)$$

Thus, one can choose

$$\begin{cases} F(x) = \min \left\{ \frac{(1+2|x|)^{\frac{1}{2}} x}{C H^2}, \frac{\ln 2}{C (1+H^2) (1+16x^2)} \right\}, \\ G(x) = \max \left\{ 2 \left( (1+2|x|)^{\frac{1}{2}} x + \frac{\ln 2}{1+16x^2} \right), 4 (1+2|x|)^{\frac{1}{2}} x \right\}, \end{cases}$$

in order to conclude that  $T^* \geq F(U_0)$  and for all  $t \in [0, T^*]$  we have:

$$U_s(t) \leq G(U_0).$$

**Remark 4.1.** Theorem 1 is just the restatement of the above result when  $r = 2$ .

**Remark 4.2.** Observe that the above arguments allow us to derive similar uniform bounds for the quantities  $\left\| (2^{js} N_j)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}$ .

## 5 Final remarks

The results of this paper generalize a part of the long time existence results for the  $abcd$  Boussinesq systems that can be found in [12]. Theorem 3 has also a homogeneous counterpart i.e. one can replace the nonhomogeneous Besov space  $B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n$  with  $(\dot{B}_{2,1}^{\frac{n}{2}} \cap \dot{B}_{2,r}^{s_1}) \times (\dot{B}_{2,1}^{\frac{n}{2}} \cap \dot{B}_{2,r}^{s_2})^n$ . For a definition and some basic properties about homogeneous Besov spaces see [3], Chapter 2. To our knowledge,  $(\dot{B}_{2,1}^{\frac{n}{2}} \cap \dot{B}_{2,r}^{s_1}) \times (\dot{B}_{2,1}^{\frac{n}{2}} \cap \dot{B}_{2,r}^{s_2})^n$  is the largest space in which one can prove long time existence for the  $abcd$  systems. Let us also point out that as opposed to classical Sobolev spaces, working with Besov spaces enables us to attain the critical regularity index  $s = \frac{n}{2} + 1$ .

### 5.1 The remaining cases

As we have seen earlier our method requires the restrictions (1.8) imposed on the parameters  $a, b, c, d$  in order to obtain the local existence theory as well as estimate (2.42), see Remark 2.1. However, the general estimations (2.16) (see Section 2.4) are valid for all the parameters (1.4) with  $b + d > 0$ . Using (2.16) and supposing that

$$1 + \varepsilon \eta_0 \geq \alpha > 0, \quad (5.1)$$

we can obtain long time existence for any solutions of (1.10) in the cases:

$$\begin{aligned} a = d = 0, \quad b > 0, \quad c < 0 \text{ and} \\ a = b = 0, \quad d > 0, \quad c < 0. \end{aligned}$$

However the Friedrichs method is not well-suited in this case because when establishing (2.16) we multiplied the frequency localized equation of  $V_j$  with  $\eta V_j$ . Thus we would run into trouble when establishing the (2.16)-type estimate for the approximations  $(\eta^m, V^m)$  introduced in Theorem 2, (3.4). Nevertheless, one can imagine another strategy in order to bypass this inconvenience namely establishing a convergence scheme for (1.10). In order to keep a certain homogeneity in this paper we will just give some brief details of how this strategy would work. Let us take  $W = 0$  to fix the ideas. Consider the linear systems:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta^{m+1} + \operatorname{div} V^{m+1} + a \varepsilon \operatorname{div} \Delta V^{m+1} + \varepsilon \operatorname{div} (\eta^m V^{m+1}) = 0, \\ (I - \varepsilon d \Delta) \partial_t V^{m+1} + \nabla \eta^{m+1} + c \varepsilon \nabla \Delta \eta^{m+1} + \varepsilon \nabla V^{m+1} V^m = 0, \\ \eta|_{t=0}^{m+1} = S_{m+1} \eta_0, \quad V|_{t=0}^{m+1} = S_{m+1} V_0, \end{cases} \quad (\mathcal{L}_{m+1})$$

with  $S_m = \sum_{j \leq m-1} \Delta_j$ , and  $(\eta^0, V^0) = (0, 0)$ . We claim that it is possible to establish the following (2.16)-type estimate:

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \int (\eta^{m+1})^2 + \varepsilon (b - c) |\nabla \eta^{m+1}|^2 + \varepsilon^2 (-c) b \nabla^2 \eta^{m+1} : \nabla^2 \eta^{m+1} \right) + \\ & \frac{1}{2} \partial_t \left( \int (1 + \varepsilon \eta^m) |V^{m+1}|^2 + \varepsilon (d - a + d \varepsilon \eta^m) |\nabla V^{m+1}|^2 + \varepsilon^2 (-a) d \nabla^2 V^{m+1} : \nabla^2 V^{m+1} \right) \\ & \leq \varepsilon P_1 \left( \|(\eta^{m+1}, V^{m+1})\|_{B_{2,r}^s}, \sqrt{\varepsilon} \|\nabla (\eta^{m+1}, V^{m+1})\|_{B_{2,r}^s}, \varepsilon \|\nabla^2 (\eta^{m+1}, V^{m+1})\|_{B_{2,r}^s} \right) \times \\ & P_2 \left( \|(\eta^m, V^m)\|_{B_{2,r}^s}, \sqrt{\varepsilon} \|\nabla (\eta^m, V^m)\|_{B_{2,r}^s}, \varepsilon \|\nabla^2 (\eta^m, V^m)\|_{B_{2,r}^s} \right), \end{aligned}$$

where  $P_1$  and  $P_2$  are two polynomials of degree 2 with coefficients depending only on the  $a, b, c, d$  parameters, on the dimension  $n$  and on the regularity index  $s$  but not on  $\varepsilon$ . Afterwards, Gronwall's

lemma combined with some stability estimates and a continuity argument would allow us to obtain long time existence for (1.10) in the remaining two cases. Of course, we would use in a decisive manner the non-cavitation condition (5.1).

## 5.2 The $abcd$ -system with smooth general topography

As we have mentioned in the introduction  $abcd$ -type models have been established in [9] for a bottom given by the surface:

$$\{(x, y, z) : z = -h + \varepsilon S(x, y)\},$$

where  $S$  is smooth enough. In this case the  $abcd$ -system reads

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + a \varepsilon \operatorname{div} \Delta V + \varepsilon \operatorname{div} ((\eta - S)V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + c \varepsilon \nabla \Delta \eta + \varepsilon \frac{1}{2} \nabla |V|^2 = 0, \\ \eta|_{t=0} = \eta_0, \quad V|_{t=0} = V_0. \end{cases} \quad (5.2)$$

We claim that our method applies to this system with some minor modifications. Let us give a few details. For the sake of simplicity we will suppose that  $\operatorname{curl} V_0 = 0$ . By localizing the above equation in Fourier space and proceeding in the same manner as in Section 2 we can see that the terms preventing us from establishing long time existence are

$$\varepsilon \int (\eta - S) \eta_j \operatorname{div} V_j \quad \text{and} \quad -c \varepsilon^2 \int (\eta - S) \nabla \operatorname{div} V_j \nabla \eta_j.$$

In order to repair this inconvenience we proceed as in Section 2.1, the only difference being that we must multiply the localized equation of  $V$  with  $(\eta - S) V_j$  rather than simply  $\eta V_j$ . Thus, we can obtain an estimate similar to (2.16) for the following quantity

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \int \eta_j^2 + \varepsilon (b - c) |\nabla \eta_j|^2 + \varepsilon^2 (-c) b (\nabla^2 \eta_j : \nabla^2 \eta_j) \right) \\ & + \frac{1}{2} \partial_t \left( \int (1 + \varepsilon (\eta - S)) V_j^2 + \varepsilon (d - a + d \varepsilon (\eta - S)) (\nabla V_j : \nabla V_j) + \varepsilon^2 (-a) d (\nabla^2 V_j : \nabla^2 V_j) \right) \leq \varepsilon G(U_s) \end{aligned}$$

where  $G$  should be some polynomial function with its coefficients not depending on  $\varepsilon$ . Afterwards proceeding like in the rest of the paper, and taking  $S \in B_{2,r}^{\tilde{s}}$  with some  $\tilde{s}$  large enough, one can obtain long time existence results for (5.2) similar to those obtained in Theorem 1.

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## 6 Appendix: Littlewood-Paley theory

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [3]. In the following if  $\Omega \subset \mathbb{R}^n$  is a domain then  $\mathcal{D}(\Omega)$  will denote the set of smooth functions on  $\Omega$  with compact support and  $\mathcal{S}$  will denote the Schwartz class of functions defined on  $\mathbb{R}^n$ . Also, we consider  $\mathcal{S}'$  is the set of tempered distributions on  $\mathbb{R}^n$ .

**Proposition 6.1.** *Let  $\mathcal{C}$  be the annulus  $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$ . There exist two radial functions  $\chi \in \mathcal{D}(B(0, 4/3))$  and  $\varphi \in \mathcal{D}(\mathcal{C})$  valued in the interval  $[0, 1]$  and such that:*

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad (6.1)$$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad (6.2)$$

$$2 \leq |j - j'| \Rightarrow \text{Supp}(\varphi(2^{-j}\cdot)) \cap \text{Supp}(\varphi(2^{-j'}\cdot)) = \emptyset \quad (6.3)$$

$$j \geq 1 \Rightarrow \text{Supp}(\chi) \cap \text{Supp}(\varphi(2^{-j}\cdot)) = \emptyset \quad (6.4)$$

the set  $\tilde{\mathcal{C}} = B(0, 2/3) + \mathcal{C}$  is an annulus and we have

$$|j - j'| \geq 5 \Rightarrow 2^j \mathcal{C} \cap 2^{j'} \tilde{\mathcal{C}} = \emptyset. \quad (6.5)$$

Also the following inequalities hold true:

$$\forall \xi \in \mathbb{R}^n, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1, \quad (6.6)$$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \quad (6.7)$$

From now on we fix two functions  $\chi$  and  $\varphi$  satisfying the assertions of the above proposition. The following two lemmas represent some basic properties of the dyadic operators.

**Lemma 6.1.** *For any  $u \in \mathcal{S}'$  we have:*

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

**Lemma 6.2.** *For any  $u \in \mathcal{S}'$  and  $v \in \mathcal{S}'$  we have that:*

$$\Delta_j \Delta_\ell u = 0 \quad \text{if } |j - \ell| \geq 2.$$

### 6.1 Besov spaces

The following propositions are a list of important basic properties of Besov spaces that are used in the paper.

**Proposition 6.2.** *A tempered distribution  $u \in \mathcal{S}'$  belongs to  $B_{2,r}^s$  if and only if there exists a sequence  $(c_j)_j$  such that  $(2^{js} c_j)_j \in \ell^r(\mathbb{Z})$  with norm 1 and an universal constant  $C > 0$  such that for any  $j \in \mathbb{Z}$  we have*

$$\|\Delta_j u\|_{L^2} \leq C c_j.$$

**Proposition 6.3.** *Let  $s, \tilde{s} \in \mathbb{R}$  and  $r, \tilde{r} \in [1, \infty]$ .*

- $B_{2,r}^s$  is a Banach space which is continuously embedded in  $\mathcal{S}'$ .
- The inclusion  $B_{2,r}^s \subset B_{2,\tilde{r}}^{\tilde{s}}$  is continuous whenever  $\tilde{s} < s$  or  $s = \tilde{s}$  and  $\tilde{r} > r$ .
- We have the following continuous inclusion  $B_{2,1}^{\frac{n}{2}} \subset \mathcal{C}_0^{14}(\subset L^\infty)$ .
- (Fatou property) If  $(u_n)_{n \in \mathbb{N}}$  is a bounded sequence of  $B_{2,r}^s$  which tends to  $u$  in  $\mathcal{S}'$  then  $u \in B_{2,r}^s$  and

$$\|u\|_{B_{2,r}^s} \leq \liminf_n \|u_n\|_{B_{2,r}^s}.$$

- If  $r < \infty$  then

$$\lim_{j \rightarrow \infty} \|u - S_j u\|_{B_{2,r}^s} = 0.$$

**Remark 6.1.** Taking advantage of the Fourier-Plancherel theorem and using 6.7 one sees that the classical Sobolev space  $H^s$  coincides with  $B_{2,2}^s$ .

**Proposition 6.4.** Let us consider  $m \in \mathbb{R}$  and a smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for all multi-index  $\alpha$ , there exists a constant  $C_\alpha$  such that:

$$\forall \xi \in \mathbb{R}^n \quad |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Then the operator  $f(D)$  is continuous from  $B_{2,r}^s$  to  $B_{2,r}^{s-m}$ .

**Proposition 6.5.** Let  $1 \leq r \leq \infty$ ,  $s \in \mathbb{R}$  and  $\varepsilon > 0$ . For all  $\phi \in \mathcal{S}$ , the map  $u \rightarrow \phi u$  is compact from  $B_{2,r}^s$  to  $B_{2,r}^{s-\varepsilon}$ .

**Proposition 6.6.** Let  $s > 0$  and  $1 \leq r \leq \infty$ . Then  $B_{2,r}^s \cap L^\infty$  is an algebra. Moreover, we have:

$$\|uv\|_{B_{2,r}^s} \lesssim \|u\|_{L^\infty} \|v\|_{B_{2,r}^s} + \|u\|_{B_{2,r}^s} \|v\|_{L^\infty}.$$

In particular, if  $s > \frac{n}{2}$  or  $s = \frac{n}{2}$  and  $r = 1$ ,  $B_{2,r}^s$  is an algebra.

We end this section with the following result concerning a commutator-type estimate. For a more general form of this result and its proof see [3] page 116, Lemma 2.100.

**Proposition 6.7.** Let us consider  $s \in \mathbb{R}$ ,  $r \in [1, \infty]$  such that  $s > 1 + \frac{n}{2}$  or  $s = 1 + \frac{n}{2}$  and  $r = 1$ . Let  $(u, v) \in B_{2,r}^s \times B_{2,r}^s$ . We denote by

$$R_j = [\Delta_j, u] \partial^\alpha v = \Delta_j (u \partial^\alpha v) - u \Delta_j \partial^\alpha v,$$

where  $\alpha$  is any multi-index with  $|\alpha| = 1$ . Then, the following estimate holds true:

$$\|(2^{js} \|R_j\|_{L^2})\|_{\ell^r(\mathbb{Z})} \lesssim \|\nabla u\|_{B_{2,r}^{s-1}} \|v\|_{B_{2,r}^s}.$$

<sup>14</sup> $\mathcal{C}_0$  is the space of continuous bounded functions which decay at infinity.

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